

# Physics 406

## Lecture #15: Models of Material Response

The constitutive relations we have assumed are naive

$$\vec{P}(\vec{r}, t) = \epsilon_0 \chi_e \vec{E}(\vec{r}, t), \quad \vec{M}(\vec{r}, t) = \chi_m \vec{H}(\vec{r}, t), \quad \vec{J}(\vec{r}, t) = \sigma \vec{E}(\vec{r}, t)$$

These implicitly assume that the response of the medium to the field is instantaneous in time.

More realistically,  $\chi_e, \chi_m, \sigma$  represent the steady-state response to forces that oscillate @ frequency  $\omega$ .

$\Rightarrow$  Local relation in the "frequency domain"

$$\vec{P}(\vec{r}, \omega) \approx \epsilon_0 \tilde{\chi}_e(\omega) \vec{E}(\vec{r}, \omega)$$

Then the ~~instantaneous~~ polarization in time

$$\vec{P}(\vec{r}, t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{P}(\vec{r}, \omega) \quad (\text{Fourier Transform})$$

$$\approx \epsilon_0 \int_{-\infty}^{\infty} d\omega \tilde{\chi}_e(\omega) \vec{E}(\vec{r}, \omega) e^{-i\omega t}$$

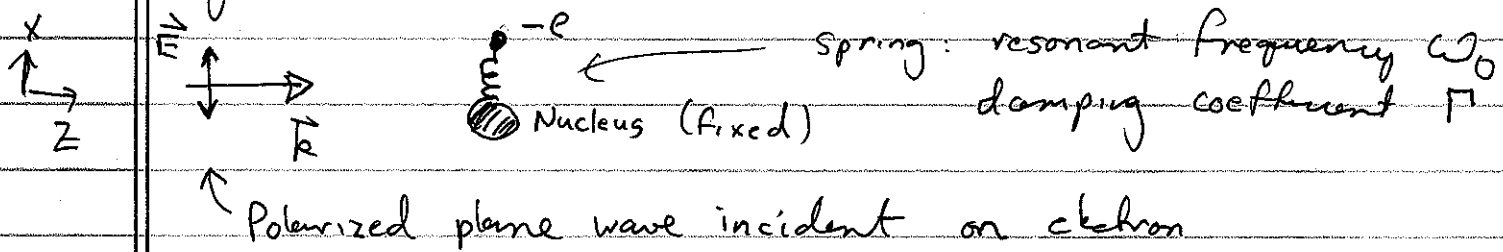
$$= \epsilon_0 \int_{-\infty}^{\infty} dt' \underbrace{\chi_e(t-t')} \vec{E}(\vec{r}, t')$$

response function

(Convolution theorem)  
More next  
Lecture

## Lorentz model of dielectrics

We can construct a simple but effective model of dielectrics by treating the bound charges in matter by charges on springs



Egm of motion:

$$m\ddot{x} = -m\omega_0^2 x - m\Gamma\dot{x} - eE(\vec{r}_{\text{electron}}, t)$$

↑ position of electron

$$\Rightarrow \text{Forced SHO: } \boxed{\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \frac{-e}{m} E(\vec{r}_e, t)}$$

Harmonic field  $E(\vec{r}_e, t) = \tilde{E} e^{-i\omega t}$

↑ complex amplitude @  $\vec{r}_{\text{electron}}$

$\Rightarrow$  Steady state response of displace of electron from equilibrium  $x(t) = \text{Re}(\tilde{x}_0 e^{-i\omega t})$

$\Rightarrow$  Use complex amplitudes and take real part in end.

$$\Rightarrow (-\omega^2 \tilde{x}_0 - i\omega\Gamma \tilde{x}_0 + \omega_0^2 \tilde{x}_0) e^{-i\omega t} = -\frac{e}{m} \tilde{E} e^{-i\omega t}$$

$$\Rightarrow \tilde{x}_0 = \frac{-e/m \tilde{E}}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

The steady state induced oscillating electric dipole moment

$$p(t) = \text{Re}(-e\tilde{x}_0 e^{-i\omega t})$$

$$\Rightarrow \tilde{p}_0 = -e\tilde{x}_0 = \frac{e^2 m}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tilde{E}$$

$\Rightarrow$  "Linear response", the <sup>complex</sup> "amplitude of the induced dipole is proportional to the electric field"

$$\tilde{p}_0 = \tilde{\alpha}(\omega) \tilde{E}$$

Where  $\tilde{\alpha}(\omega) = \frac{e^2}{m} \left( \frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma} \right) =$  Atomic "dynamic polarizability"

Meaning of complex polarization

$$p(t) = \text{Re}(\tilde{\alpha} \tilde{E} e^{-i\omega t}), \text{ let } \tilde{E} = |E| e^{i\phi}$$

$$= \text{Re}(\tilde{\alpha} e^{-i(\omega t - \phi)}) |E|$$

$$= \underbrace{\text{Re}(\tilde{\alpha}) |E| \cos(\omega t - \phi)}_{\text{In phase with } E} + \underbrace{\text{Im}(\tilde{\alpha}) |E| \sin(\omega t - \phi)}_{\text{In quadrature with } E}$$

In phase  
with  $E$

In quadrature  
with  $E$

The  $\text{Im}(\tilde{\alpha})$  describes how much energy is absorbed by molecule in steady state as shown in previous lectures.

Near resonance behavior  $\omega \approx \omega_0$

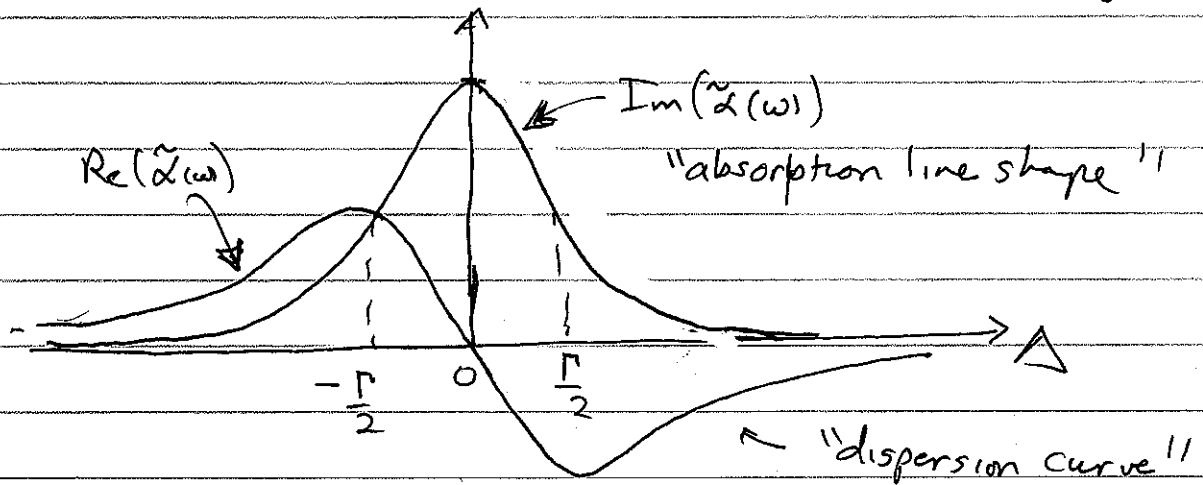
Define "detuning"  $\Delta \equiv \omega - \omega_0$

We showed in lecture, when  $\Delta, \Gamma \ll \omega, \omega_0$

$$\tilde{\alpha}(\omega) \approx \left( \frac{e^2}{2m\omega_0} \right) \left( \frac{-1}{\Delta + i\frac{\Gamma}{2}} \right) \quad \text{"Complex Lorentzian"}$$

$$= \frac{e^2}{2m\omega_0} \left[ \frac{-\Delta}{\Delta^2 + \frac{\Gamma^2}{4}} + i \left( \frac{\frac{\Gamma}{2}}{\Delta^2 + \frac{\Gamma^2}{4}} \right) \right]$$

↑  
Lorentzian



The "line width" of the resonance is  $\Gamma$  (FWHM)

The real part of  $\alpha$  gives rise to "dispersion" as we will see next lecture.

## Suceptibility in a dilute gas

For a low density medium, the field seen by a molecule is only due to the external field, and not the field of the neighboring molecules. In condensed matter this is a poor approximation - there are so-called "local field corrections"

In the absence of local-field corrections, given  $N$  molecules / Volume, each with  $Z$  electrons/molecule

There can be multiple resonances

each with parameters  $\omega_j, \Gamma_j$ ;  $f_j$  electrons / molecule

Contribute to resonance ( $f_j \equiv$  "oscillator strength")

$$\sum_j f_j = Z \quad (\text{sum rule})$$

Total polarization density

$$\tilde{P}(\omega) = \sum_j N f_j \tilde{P}_j(\omega)$$

$$= \frac{Ne^2}{m} \sum_j \frac{f_j \tilde{E}(\omega)}{\omega_j^2 - \omega^2 - i\omega\Gamma_j} = \epsilon_0 \tilde{\chi}(\omega) \tilde{E}(\omega)$$

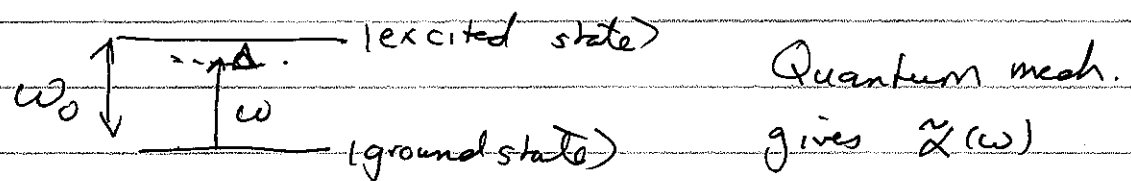
$$\Rightarrow \tilde{\chi}(\omega) = \sum_j \frac{f_j \omega_p^2}{\omega_j^2 - \omega^2 - i\omega\Gamma_j} \quad \text{where}$$

$$\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}$$

"Plasma freq"

Aside: "Oscillator strength  $f_j$ ",  $\sum_j f_j = Z$

- Actually, in classical physics  $f_j$  is a "fudge factor"
- $f_j$  really arises because the quantum mechanical response is different from classical spring force



Complex dielectric coefficient

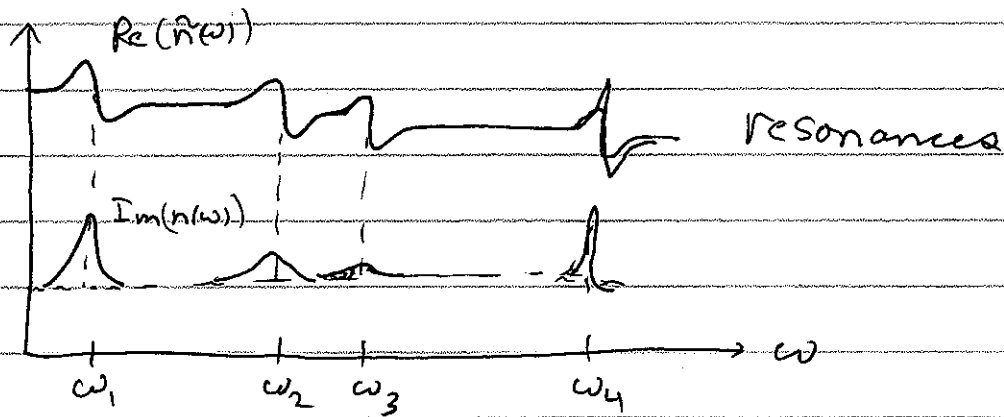
$$\begin{aligned}\tilde{\kappa}_e(\omega) &= 1 + \tilde{\chi}_e(\omega) = 1 + \frac{N}{\epsilon_0} \tilde{\chi}(\omega) \\ &= 1 + \sum_j f_j \frac{\omega_p^2}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}\end{aligned}$$

$\Rightarrow$  Complex index of refraction (with  $\mu = \mu_0$ )

$$\tilde{n}(\omega) = \sqrt{\tilde{\kappa}_e(\omega)} = \underbrace{n_R(\omega)}_{\text{real part}} + i \underbrace{n_I(\omega)}_{\text{imaginary part}}$$

For dilute gas  $|\tilde{\chi}| \ll 1 \Rightarrow \tilde{n}(\omega) \approx 1 + \frac{1}{2} \tilde{\chi}_e(\omega)$

$$\begin{aligned}n_R(\omega) &\approx 1 + \sum_j f_j \frac{\omega_p^2 (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\omega\Gamma_j)^2} \\ n_I(\omega) &\approx \sum_j f_j \frac{\omega_p^2 \omega \Gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\Gamma_j)^2}\end{aligned}$$



Outside of resonance, medium is not absorbing (transparent if smooth and non-scattering).

There, can neglect  $\Gamma_j$  terms in denominator

$$\Rightarrow n_R(\omega) \approx 1 + \frac{1}{2} \sum_j f_j \frac{\omega_p^2}{\omega_j^2 - \omega^2}$$

For optical frequencies, in transparent materials  $\omega_j$  in ultra-violet  $\Rightarrow \omega \ll \omega_j \quad \forall j$  of interest

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2} \left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \approx \frac{1}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

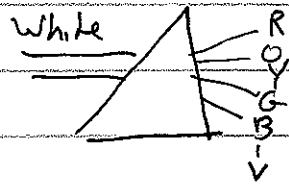
$$\Rightarrow n_R(\omega) \approx 1 + \left(\frac{1}{2} \sum_j f_j \frac{\omega_p^2}{\omega_j^2}\right) + \omega^2 \left(\frac{1}{2} \sum_j f_j \frac{\omega_p^2}{\omega_j^4}\right)$$

Or in terms of  $\lambda_* = \frac{2\pi c}{\omega}$  (vacuum wavelength)

$$n_R(\lambda) \approx 1 + A \left(1 + \frac{B}{\lambda^2}\right)$$

Cauchy's formula

Note: In the transparent bands, the index of refraction (real) increases with increasing  $\omega$  (shorter wave length). This is known as "normal dispersion". Slow increase with  $\frac{1}{\lambda}$



"Dispersion in a prism"  
Normal

Around resonance, index goes through a rapid rise and then fall.



$n$  decreases with increasing  $\omega$

⇒ "Anomalous Dispersion"

Note: in this region medium is also highly absorbing.

Thus, for typical medium we do not see anomalous dispersion unless it is "engineered" is a special material.



## Propagation in absorbing, dispersive medium

Complex wave number:  $\tilde{k} = \tilde{n}(\omega) \frac{\omega}{c}$

$$\Rightarrow \tilde{k} = k_R + i k_I$$

•  $k_R = n_R(\omega) \frac{\omega}{c} = \frac{\omega}{v_{\text{phase}}(\omega)}$  "Dispersion relation"

$$v_{\text{phase}}(\omega) = \frac{c}{n_R(\omega)} : \text{Frequency dependent phase velocity}$$

Note: In regions of anomalous dispersion

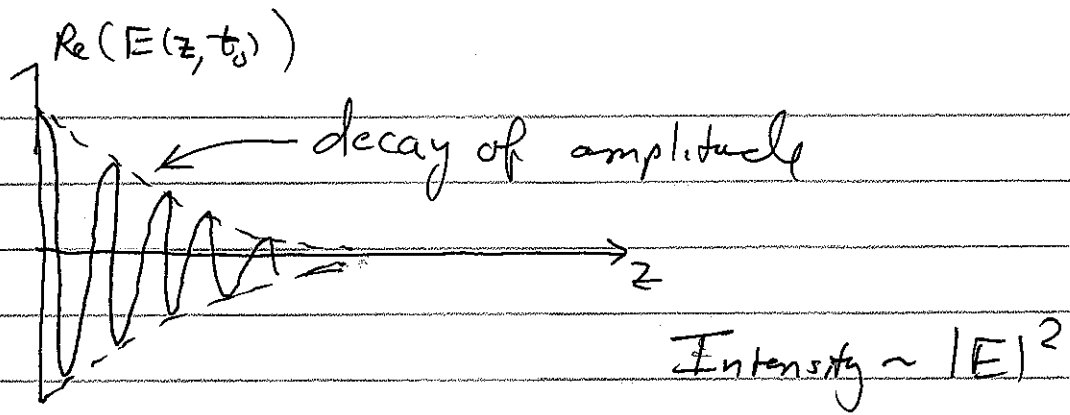
$$n_R(\omega) < 1 \Rightarrow v_{\text{phase}}(\omega) > c \quad \nabla \quad ?$$

This is not a violation of special relativity.  
As we will discuss no information travels at  $v_{\text{phase}}$ .

•  $k_I = n_I(\omega) \frac{\omega}{c}$

$$E(z, t) = \underline{E} e^{i(kz - \omega t)} = \underline{E} e^{-k_I z} e^{i(k_R z - \omega t)}$$

$k_I$  = attenuation coefficient



Absorption of power ~~rate~~ at spatial rate

$$\beta = \underbrace{2k_I}_{\text{from square of amplitude}} = 2n_I(\omega) \frac{\omega}{c}$$

Attenuation through absorption:

Rate of change of intensity with propagation

$$\frac{dI(z)}{dz} = \frac{d}{dz} (e^{-\beta z} I(0))$$

$$\Rightarrow \boxed{\frac{dI(z)}{dz} = -\beta I} \quad \text{"Beer's Law"}$$

$$\boxed{\text{"Beer's length"} = \frac{1}{\beta} = \frac{I_{vac}}{\pi n_I(\omega) I}}$$

## Conductors : Drude Model

In conductor, carriers of charge are "free". Good models really require the quantum description of band structure. We can arrive at a qualitative picture with a classical free electron gas with "friction" due to collisions. This is known as the "Drude model".

Eqs of motion for electron:

$$m_e \frac{d\vec{v}_e}{dt} = -m\gamma \vec{v}_e - e\vec{E}(t)$$

↖ Friction due to collisions

In "frequency domain"

$$-i\omega m \vec{v}_e = -m\gamma \vec{v}_e - e\vec{E}$$

$$\Rightarrow \vec{v}_e = \left( \frac{-e}{m} \right) \frac{\vec{E}}{-i\omega + \gamma}$$

$$\Rightarrow \vec{J} = -Ne \vec{v}_e = \frac{Ne^2}{m} \frac{\vec{E}}{-i\omega + \gamma} = \sigma(\omega) \vec{E}$$

Freq dependent conductivity

$$\sigma(\omega) = \epsilon_0 \frac{\omega_p^2}{-i\omega + \gamma}$$

Note: When  $\omega \ll \gamma$

$$\sigma \rightarrow \epsilon_0 \frac{\omega_p^2}{\gamma} = \frac{Ne^2}{m\gamma} = \text{D.C. Conductivity}$$

When  $\omega \gg \gamma$

$$\sigma \rightarrow i \epsilon_0 \frac{\omega_p^2}{\omega} : \text{Pure imaginary}$$

$$\Rightarrow \tilde{k} = \mu_0 \omega^2 - \mu_0 \epsilon_0 \omega_p^2 = \frac{\omega^2 - \omega_p^2}{c^2}$$

$$\Rightarrow \tilde{k} \text{ pure real ("Plasma" see homework)}$$

At high frequencies, no collisions

$\Rightarrow$  No attenuation