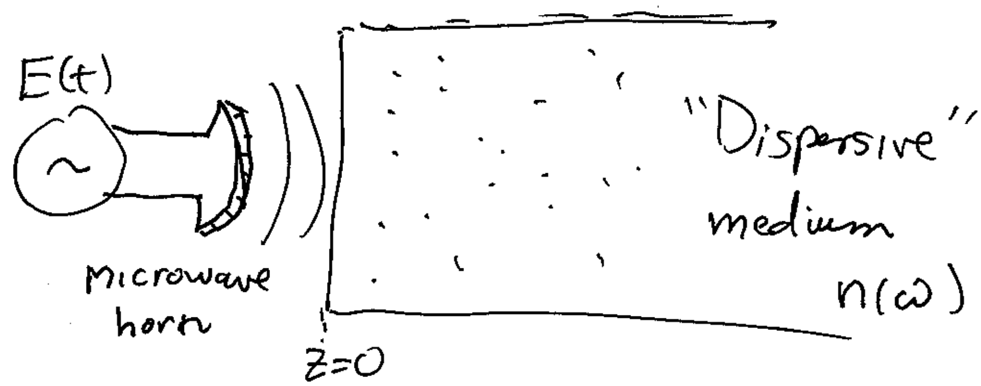
# Physics 406

## Lecture 17: Propagation and Dispersion

Suppose we inject a signal $\vec{E}(t)$ at the input face of some medium (semi-infinite)




The input signal can be Fourier transformed

$$E(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\, e^{-i\omega t} = E(z=0, t)$$

(Boundary value)

To each frequency we assign a wave number according to the dispersion relation

$$k(\omega) = \frac{\omega}{c} n(\omega) = \frac{\omega}{v_p(\omega)}$$

frequency dependent index of refraction

$$E(z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\, e^{-i\omega\left(t - \frac{z}{v_p}\right)}$$

$$= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\, e^{i(k(\omega) z - \omega t)}$$

Consider just two frequencies

$$E(z,t) = \mathrm{Re}\left(E_1\, e^{i(k_1 z - \omega_1 t)} + E_2\, e^{i(k_2 z - \omega_2 t)}\right)$$

suppose $E_1 = E_2 \equiv E_0$

$$= \mathrm{Re}\left(E_0 \left(e^{i\theta_1} + e^{i\theta_2}\right)\right)$$

where $\theta_i = k_i z - \omega_i t \qquad i = 1, 2$

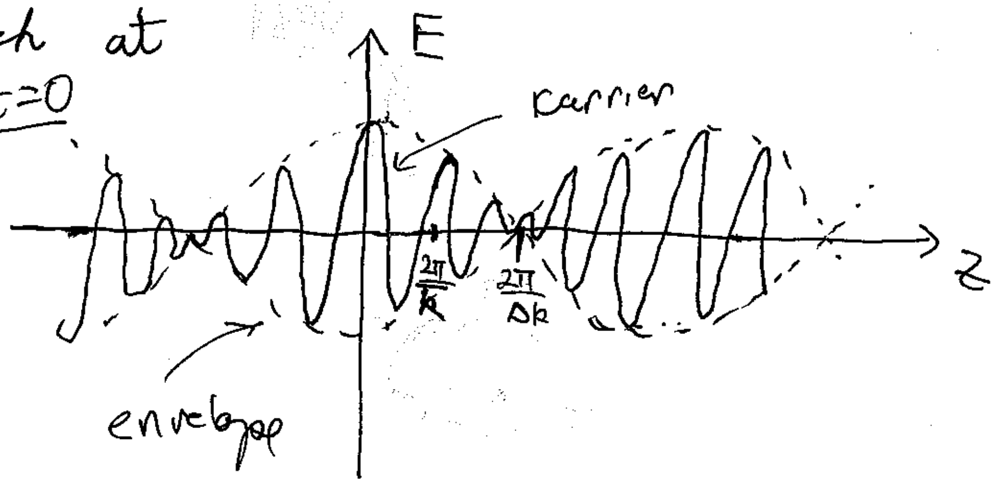
Aside: $$\left[ e^{i\theta_1} + e^{i\theta_2} = e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\left[e^{i\left(\frac{\theta_1-\theta_2}{2}\right)} + e^{-i\left(\frac{\theta_1-\theta_2}{2}\right)}\right] \right.$$

$$\left. = e^{i\bar{\theta}}\, 2\cos(\Delta\theta) \quad \text{where} \quad \bar{\theta} = \frac{\theta_1+\theta_2}{2}, \quad \Delta\theta = \frac{\theta_1-\theta_2}{2} \right]$$

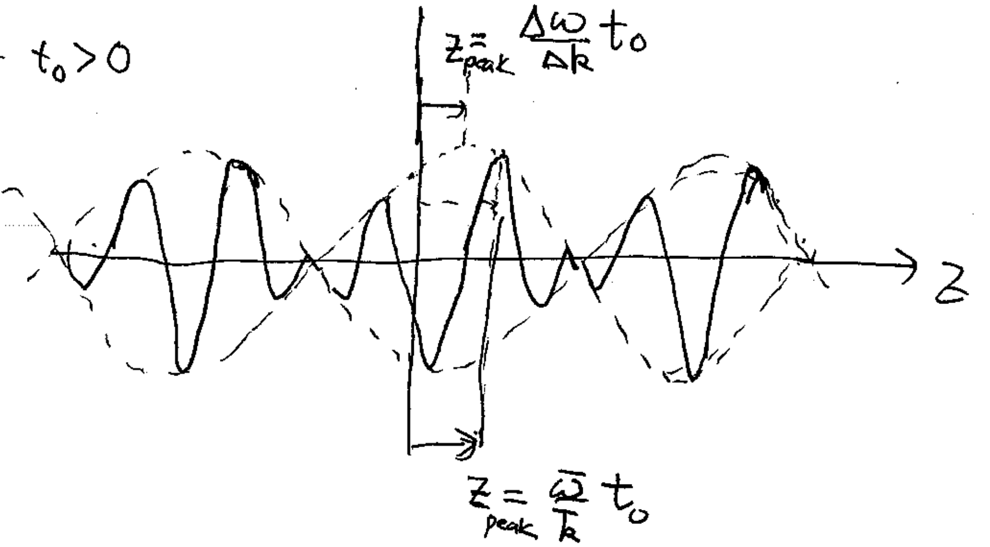
$$\Rightarrow E(z,t) = E_0 \cos(\bar{k} z - \bar{\omega} t)\cos(\Delta k z - \Delta\omega t)$$

where $\bar{k} = \frac{k_1+k_2}{2} \qquad \Delta k = \frac{k_1-k_2}{2}$ etc.

Sketch at $\underline{t=0}$ $\qquad k_1 > k_2$



At $t_0 > 0$

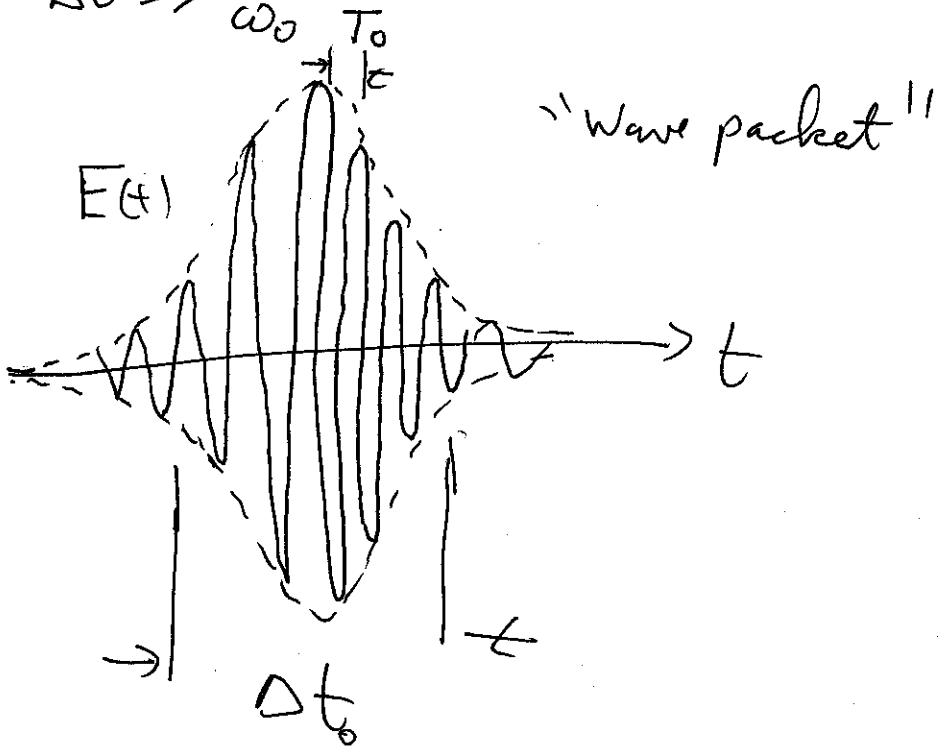



$\Rightarrow$ Velocity of carrier wave different than velocity of envelope

## Case of monochromatic pulse

Example: Gaussian pulse $E(t) = \underbrace{E_0\, e^{-\frac{t^2}{2(\Delta t_0)^2}}}_{\mathcal{E}(t) \leftarrow \text{envelope}} \cos(\omega_0 t)$

Take $\Delta t \gg \frac{2\pi}{\omega_0}$



Fourier Transform

$$\tilde{E}(\omega) = \frac{1}{2}\tilde{\mathcal{E}}(\omega-\omega_0) + \frac{1}{2}\tilde{\mathcal{E}}(\omega+\omega_0)$$

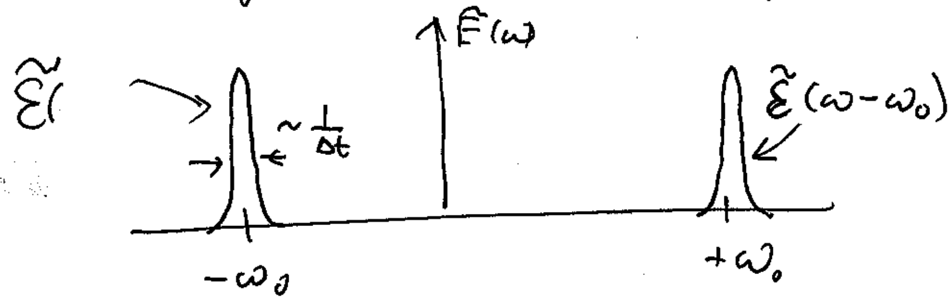
where $\tilde{\mathcal{E}}(\omega) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathcal{E}(t)\,e^{-i\omega t}$ : Envelope fourier transform

($\mathcal{E}(t)$ ← envelope function)

⇒ Propagating pulse:

$$E(z,t) = \frac{1}{2}\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega-\omega_0)\,e^{i(k(\omega)z-\omega t)} + \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega+\omega_0)\,e^{i(k(\omega)z-\omega t)}\right]$$

Now, for quasi-monochromatic pulses




$$\Rightarrow \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega-\omega_0)\,e^{i(k(\omega)z-\omega t)} \cong \int_{0}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega-\omega_0)\,e^{i(k(\omega)z-\omega t)}$$

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega+\omega_0)\,e^{i(k(\omega)z-\omega t)} \cong \int_{-\infty}^{0}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega+\omega_0)\,e^{i(k(\omega)z-\omega t)}$$

$$= \int_{0}^{\infty}\frac{d\omega'}{2\pi}\,\tilde{\mathcal{E}}(-\omega'+\omega_0)\,e^{i(k(-\omega')z+\omega' t)} \quad \begin{pmatrix}\text{where}\\ \omega' = -\omega\end{pmatrix}$$

Aside: $k(-\omega) = -\frac{\omega}{c} n(-\omega) = -\frac{\omega}{c} n(\omega) = -k(\omega)$

Since $n(-\omega) = n(\omega)$ (one can show from causality)

Let $\Omega \equiv \omega - \omega_0$

$$\Rightarrow E(z,t) \cong \frac{1}{2}\int_{-\omega_0 \to -\infty}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(\Omega)\, e^{i[k(\Omega+\omega_0)z - \Omega t]}\, e^{-i\omega_0 t}$$

$$+ \frac{1}{2}\int_{-\omega_0}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(-\Omega)\, e^{-i[k(\Omega+\omega_0)z - \Omega t]}\, e^{+i\omega_0 t}$$

But $\tilde{\mathcal{E}}(-\Omega) = \tilde{\mathcal{E}}^*(\Omega)$

$$\Rightarrow E(z,t) \cong \mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(\Omega)\, e^{i[k(\Omega+\omega_0)z - \Omega t]}\, e^{-i\omega_0 t}\right]$$

<u>Now</u>: Since $\tilde{\mathcal{E}}(\Omega)$ is a very narrow function centered around $\Omega = 0$, we can expand $k(\Omega+\omega_0)$ around $\Omega = 0$

$$k(\Omega+\omega_0) = \underbrace{k(\omega_0)}_{\equiv k_0} + \Omega \left.\frac{dk}{d\omega}\right|_{\omega_0} + \frac{1}{2}\Omega^2 \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0} + \cdots$$

<u>Taylor series</u>

Keeping only the first two terms:

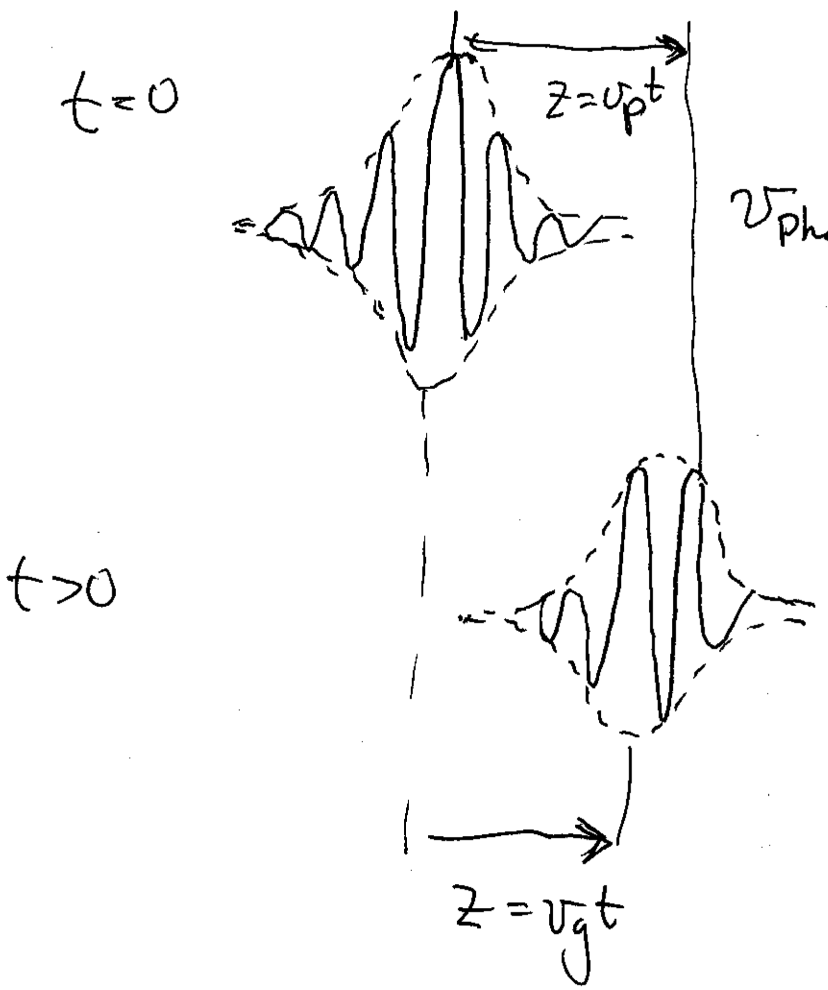
$$E(z,t) \cong \mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_0} z\right)}\right] e^{i(k_0 z - \omega_0 t)}$$

Define group velocity: $\boxed{\left.\frac{d\omega}{dk}\right|_{\omega_0} \equiv v_g}$

$$\Rightarrow E(z,t) = \mathrm{Re}\left[\mathcal{E}\left(t - \frac{z}{v_g}\right) e^{i(k_0 z - \omega_0 t)}\right]$$

For our example of a Gaussian

$$E(z,t) = E_0\, e^{-\frac{\left(t - \frac{z}{v_g}\right)^2}{2\Delta t_0^2}} \cos(k_0 z - \omega_0 t)$$




$$v_{\text{phase}} = \frac{\omega_0}{k_0}$$

Pulse peak travels at $v_g$

Effect of higher order terms in Taylor series expansion of $k(\omega)$: <u>Dispersion</u>

$$k'' \equiv \frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = \frac{dv_g}{d\omega}\left(\frac{1}{-v_g^2}\right)$$

: Group velocity ~~dypesi~~ dispersion (GVD)

Due to the spread of frequencies in the pulse $\Delta\omega$ there will be a spread of group velocities $\Delta v_g = \frac{dv_g}{d\omega}\Delta\omega$

$\Longrightarrow$ Spread in the arrival times of peak after propagating a distance $z$.

$$\Delta t_{arrive}(z) = \Delta\left(\frac{z}{v_g}\right) = z\left(-\frac{\Delta v_g}{v_g^2}\right)$$

$$= -z\left(\frac{\frac{dv_g}{d\omega}\Delta\omega}{v_g^2}\right) = -k'' z\,\Delta\omega$$

But $\Delta\omega \sim \frac{1}{\Delta t_0}$ (Fourier duality)

$$\Longrightarrow \Delta t_{arrive}(z) \cong -\frac{k'' z}{\Delta t^2}$$

Add variance:

$$\boxed{\Delta t_{total}^2(z) = \Delta t_0^2 + \Delta t_{arrive}^2(z) = \Delta t_0^2 + \frac{k''}{\Delta t_0^2}z^2}$$

$\Rightarrow$ Pulse <u>disperses</u> (spreads out) in a medium with non-zero $k''$