Physics 406

Lecture 18: Potential formulation of Maxwell's Equations and Retarded-time Solutions.

We have shown that Maxwell's Equations have electromagnetic wave solutions in regions of vacuum. But, where do these wave come from?

Answers: Sources $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$.

Let us recall the formal solution in <u>statics</u> $\frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{B}}{\partial t} = 0$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Electromagnetic potentials:

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$$

electrostatic potential

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$

vector potential

$V(\vec{r}) \to V(\vec{r}) + V_0$, E unchanged (<u>Ground</u>)

$\vec{A}(\vec{r}) \to \vec{A}(\vec{r}) + \vec{\nabla}\chi(\vec{r})$ $\vec{B}$ unchanged (<u>Gauge transform</u>)

arbitrary scalar field

$\vec{\nabla}\cdot\vec{A}$ irrelevant to physical force

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla} V) = \rho/\epsilon_0$$

$$\Rightarrow \boxed{\nabla^2 V = -\rho/\epsilon_0} \quad \text{Poisson's Eqn.}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Magnetostatic gauge $\vec{\nabla}\cdot\vec{A} = 0$

$$\Rightarrow \boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

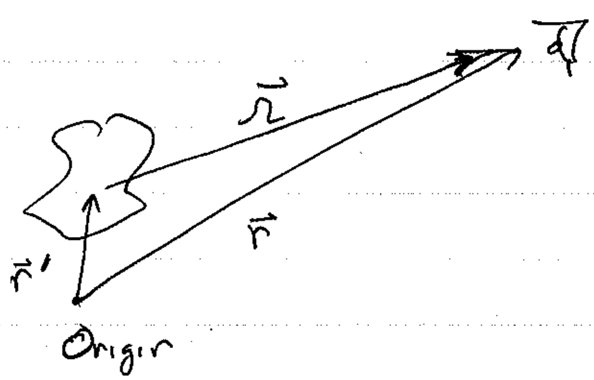
Each Cartesian component of $\vec{A}$ satisfies Poisson Eqn.

Solution with $V, \vec{A} \to 0$ @ ∞

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \equiv \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{\mathscr{r}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\int d^3r \frac{\vec{J}(\vec{r}')}{\mathscr{r}}$$

where $\mathscr{r} = |\vec{r}-\vec{r}'|$



$\Rightarrow$ In statics

- $\vec{E}(\vec{r}) = -\vec{\nabla} V = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(\vec{r}\,') \left(-\vec{\nabla}\frac{1}{\mathscr{r}}\right)$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(\vec{r}\,') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \underline{\text{Coulomb's Law}}$$

- $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}\,') \times \left(\vec{\nabla}\frac{1}{\mathscr{r}}\right)$

$$= \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \underline{\text{Biot - Savart Law}}$$

Potential formulation in Electrodynamics

Maxwell's Eqns (Microscopic Version)

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

- Since $\vec{\nabla}\cdot\vec{B} = 0 \Longrightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ with Gauge Invariance

Now by Faraday, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t}\right)$

$$\Rightarrow \quad \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \quad \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\therefore \quad \boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A}}$$

Gauge tranformation:

$$\boxed{\vec{A} \Rightarrow \vec{A} + \vec{\nabla}\chi} \quad \begin{cases} \vec{B} \Rightarrow \vec{B} \\ \vec{E} \Rightarrow \vec{E} - \vec{\nabla}\frac{\partial \chi}{\partial t} \end{cases}$$

$\Rightarrow$ To keep $\vec{E}$ the same, must also transform V

$$\boxed{V \Rightarrow V - \frac{\partial \chi}{\partial t}} \quad \text{Then } \vec{E} \Rightarrow \vec{E}$$

The source-free Maxwell's Eqn's define potentials.
Now plug these into Eqn's with ρ and J

- $$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\epsilon_0$$

$$\Rightarrow \boxed{+\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0}$$

- $$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{A} - \mu_0\epsilon_0\vec{\nabla}\left(+\frac{\partial V}{\partial t}\right)$$

$$\Rightarrow \underline{\left(\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right)}$$

Choice of Gauge? Again, $\vec{\nabla}\cdot\vec{A}$ is arbitrary

Eg. "Lorentz Gauge" (for reasons that will become clear later)

$$\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0}$$

We then have a symmetric form for V and $\vec{A}$

$$\boxed{\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V(\vec{r},t) &= -\rho(\vec{r},t)/\epsilon_0 \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) &= -\mu_0\vec{J}(\vec{r},t)\end{aligned}}$$

In the Lorentz Gauge, V and $\vec{A}$ satisfy the wave equation with source terms

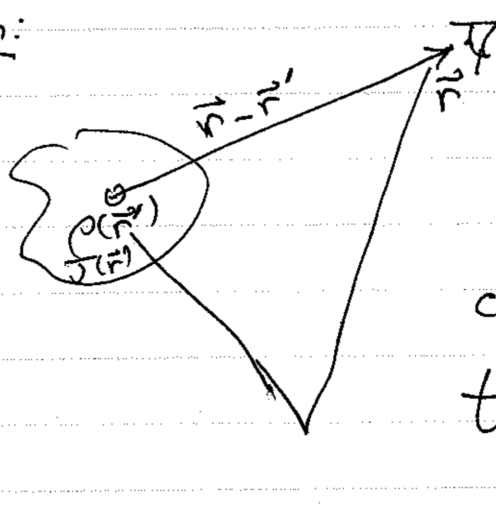
Solution with no sources @ ∞

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,', t_{ret})}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t_{ret})}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

where $t_{ret} \equiv t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ "retarded time"

Intuition:



Potential @ $\vec{r}, t$ dependence on what source what sources were doing @ an earlier (retarded) time $t -$ (time light takes to get from $\vec{r}\,'$ to $\vec{r}$)

Important example: Point Source

$$\rho(\vec{r},t) = a\,\delta(\vec{r}-\vec{r}_0)\,\delta(t-t_0)$$

Like a ~~sh~~ dropping a pebble in a pond

- Short impulse @ time $t = t_0$ localized @ point $\vec{r} = \vec{r}_0$

$$\Rightarrow V(\vec{r},t) = \frac{a}{4\pi\epsilon_0} \int d^3r' \, \frac{\delta(\vec{r}'-\vec{r}_0)\,\delta\left(t - \frac{|\vec{r}-\vec{r}'|}{c} - t_0\right)}{|\vec{r}-\vec{r}'|}$$

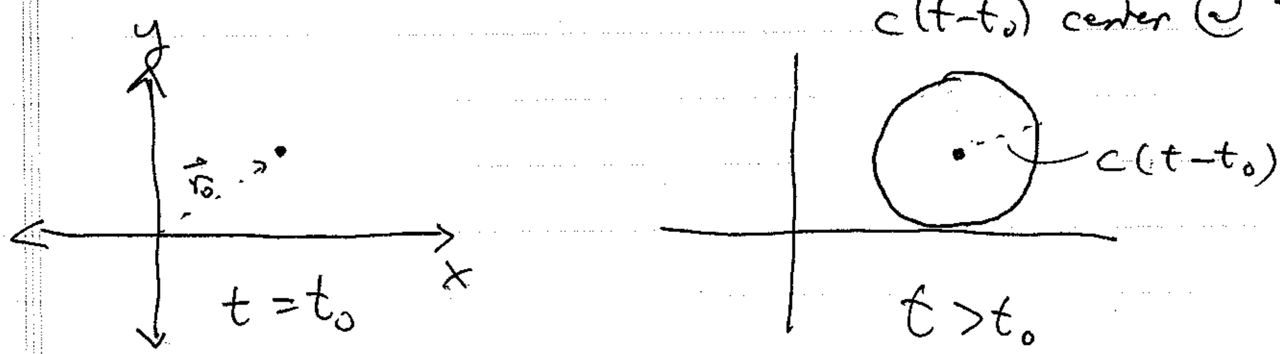
$$\Rightarrow \boxed{V(\vec{r},t) = \frac{a}{4\pi\epsilon_0} \, \frac{\delta\left(t - \left(t_0 + \frac{|\vec{r}-\vec{r}_0|}{c}\right)\right)}{|\vec{r}-\vec{r}_0|}}$$

Concentrated on a shell in space and time

$V(\vec{r},t)$ non-zero only when

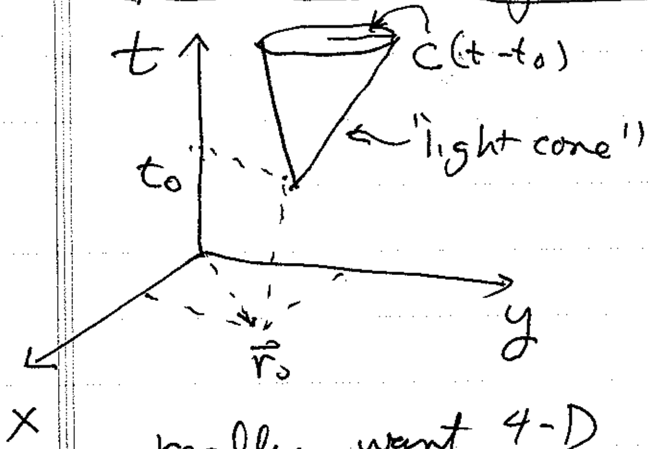
$|\vec{r}-\vec{r}_0| = c(t-t_0)$ = Sphere of radius $c(t-t_0)$ center @ $\vec{r}_0$

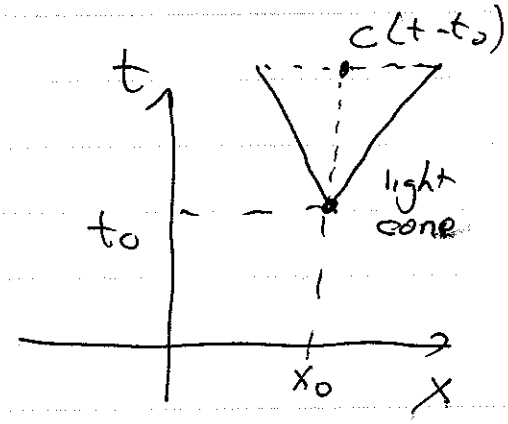
Sliced in x-y plane



Spherical wave front expanding from $\vec{r}_0$

Space-time diagram





really want 4-D diagram, but hard enough to draw 3-D

2D projection

Harmonically oscillating Sources

Consider the case (take real part in end)

$$\begin{cases} \rho(\vec{r},t) = \tilde{\rho}(\vec{r})\, e^{-i\omega t} \\ \vec{J}(\vec{r},t) = \tilde{\vec{J}}(\vec{r})\, e^{-i\omega t} \end{cases}$$

Then $V, \vec{A}$ will oscillates as $e^{-i\omega t}$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int d^3r'\, \tilde{\rho}(\vec{r}\,')\, \frac{e^{-i\omega\left(t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r},t) = \mu_0 \int d^3r'\, \tilde{\vec{J}}(\vec{r}\,')\, \frac{e^{-i\omega\left(t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}}{|\vec{r}-\vec{r}\,'|}$$

Example: Osicalling point $\tilde{\rho}(\vec{r}) = a\,\delta(\vec{r}-\vec{r}_0)$

$$\Rightarrow \boxed{V(\vec{r},t) = \frac{a}{4\pi\epsilon_0} \frac{e^{i(k|\vec{r}-\vec{r}_0| - \omega t)}}{|\vec{r}-\vec{r}_0|}, \quad k = \frac{\omega}{c}}$$

Spherical wave eminating from $\vec{r}_0$

Then we have a symmetric form for V and $\vec{A}$

$$\boxed{\begin{aligned} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V(\vec{r},t) &= -\rho(\vec{r},t)/\epsilon_0 \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \vec{A}(\vec{r},t) &= -\mu_0 \vec{J}(\vec{r},t) \end{aligned}}$$

In Lorentz Gauge V and $\vec{A}$ satisfy wave equation with source terms

Harmonic Solutions

Let us consider harmonically oscillating sources

$$\rho(\vec{r},t) = \mathrm{Re}\left(\tilde{\rho}(\vec{r})\, e^{-i\omega t}\right)$$

$$\vec{J}(\vec{r},t) = \mathrm{Re}\left(\underset{\sim}{\vec{J}}(\vec{r})\, e^{-i\omega t}\right)$$

Then V and $\vec{A}$ oscillate harmonically

(Of course, a more general ~~[illegible]~~ source ~~[illegible]~~ can be decomposed into harmonic components via a Fourier transform)

$$\Rightarrow \boxed{\begin{aligned} \left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{V}(\vec{r}) &= -\tilde{\rho}(\vec{r})/\epsilon_0 \\ \left(\nabla^2 + \frac{\omega^2}{c^2}\right) \underset{\sim}{\vec{A}}(\vec{r}) &= -\mu_0 \underset{\sim}{\vec{J}}(\vec{r}) \end{aligned}} \quad \text{Helmholtz Eqn.}$$