Physics 406

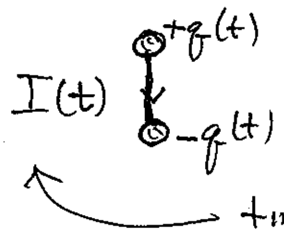
Lecture 19: Electric Dipole Radiation

The most fundamental example of electromagnetic radiation is an oscillating electric dipole.

Consider a "tiny antenna" consisting of two tiny (point-like) metal spheres connected by a short wire of length s

Through some process:

$$q(t) = q_0 \cos\omega t$$



$\Rightarrow$ Dipole $p(t) = q(t)\,s$

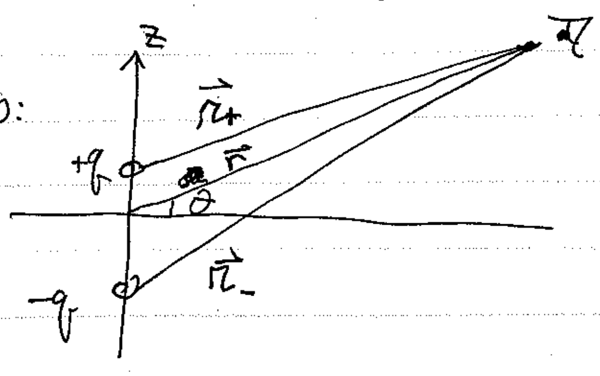
time-dependent current oscillating between terminals $I(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin\omega t$

In complex notation:

$$q(t) = \mathrm{Re}(q_0 e^{-i\omega t}) \qquad p(t) = \mathrm{Re}(\underbrace{q_0 s}_{p_0}\, e^{-i\omega t})$$

$$I(t) = \mathrm{Re}(\underbrace{-i\omega q_0}_{\tilde{I}_0}\, e^{-i\omega t})$$

Potentials:



$$V(\vec{r}, t) = \mathrm{Re}(\tilde{V}(\vec{r})\, e^{-i\omega t})$$

$$\vec{A}(\vec{r}, t) = \mathrm{Re}(\tilde{\vec{A}}(\vec{r})\, e^{-i\omega t})$$

Scalar Potential

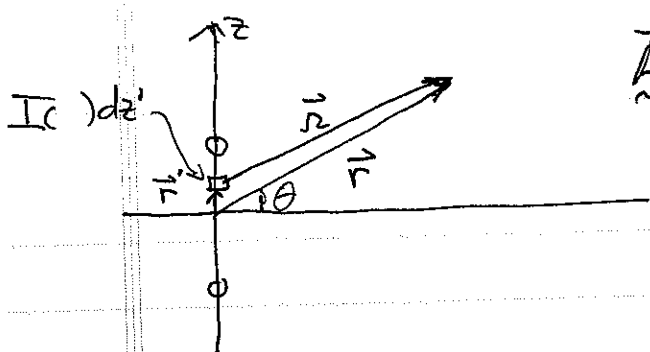
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q(t-\frac{\mathscr{R}_+}{c})}{\mathscr{R}_+} - q\frac{(t-\frac{\mathscr{R}_-}{c})}{\mathscr{R}_-} \right\}$$

$$\Rightarrow \underset{\sim}{V}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{ q_0\frac{e^{-i\omega(t-\frac{\mathscr{R}_+}{c})}}{\mathscr{R}_+} - q_0\frac{e^{-i\omega(t-\frac{\mathscr{R}_-}{c})}}{\mathscr{R}_-} \right\}$$

$$\Rightarrow \boxed{\underset{\sim}{V}(\vec{r}) = \frac{q_0}{4\pi\epsilon_0}\left\{ \frac{e^{ik\mathscr{R}_+}}{\mathscr{R}_+} - \frac{e^{ik\mathscr{R}_-}}{\mathscr{R}_-} \right\}, \quad k = \frac{\omega}{c}}$$

Superposition of two spherical waves eminating from charged points

Vector Potential



$$\underset{\sim}{\vec{A}}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\underset{\sim}{\vec{J}}(\vec{r}',t-\frac{\mathscr{R}}{c})}{\mathscr{R}}$$

$$= \frac{\mu_0}{4\pi}\int_{-s/2}^{s/2} \frac{\tilde{I}(t-\frac{\mathscr{R}}{c})\,dz'}{\mathscr{R}}\,\hat{z}$$

$$\Rightarrow \underset{\sim}{\vec{A}}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-s/2}^{s/2} \tilde{I}_0\frac{e^{-i\omega(t-\frac{\mathscr{R}}{c})}}{\mathscr{R}}\,dz'\,\hat{z}$$

$$\Rightarrow \boxed{\underset{\sim}{\vec{A}}(\vec{r}) = \frac{\mu_0}{4\pi}\tilde{I}_0\int_{-s/2}^{s/2} \frac{e^{ik\mathscr{R}}}{\mathscr{R}}\,dz'\,\hat{z}}$$

Approximation #1: Multipole Expansion

As in statics, ~~the~~ for a localized distribution, when observing far from source, the potentials decompose into ~~[illegible]~~ multipoles

$$r >> s \;, \quad \frac{s}{r} << 1 \qquad \forall z' \;\; \frac{z'}{r} << 1$$

$$\Rightarrow \mathscr{r} = \sqrt{r^2 - 2rz'\cos\theta + (z')^2} = r\sqrt{1 - 2\left(\frac{z'}{r}\right)\cos\theta + \left(\frac{z'}{r}\right)^2}$$

$$\approx r\left(1 - \frac{z'}{r}\cos\theta\right)$$

$$\Rightarrow \underset{\sim}{\vec{A}}(\vec{r}) = \frac{\mu_0}{4\pi}\underset{\sim}{I_0}\frac{e^{ikr}}{r}\int_{-s/2}^{s/2}\frac{e^{-ikz'\cos\theta}}{1 - \frac{z'}{r}\cos\theta}\,dz'\,\hat{z}$$

Approximation #2: Nonrelativistic motion

Speed of charges: $\frac{s}{T}$, $\quad T = \frac{2\pi}{\omega}$ (period of oscillation)

$$\Rightarrow \frac{s}{T} << c \quad \text{or} \quad s << cT \quad \text{or} \quad T >> \frac{s}{c}$$

Time it take light to transverse source is tiny compared to time over which sources change.

Yet another way to say this

$$\frac{c}{\nu} >> s \quad \Longrightarrow \quad \boxed{\lambda >> s}$$

(wavelength of radiation large compared to size of source)

$$\Longrightarrow \boxed{ks << 1}$$

Thus:

small phase

$$\frac{e^{-ikz'\cos\theta}}{1-\frac{z'}{r}\cos\theta} \cong \frac{1-ikz'\cos\theta}{1-\frac{z'}{r}\cos\theta} \approx (1-ikz'\cos\theta)\left(1+\frac{z'}{r}\cos\theta\right)$$

$$\cong 1 + \left(-ik+\frac{1}{r}\right) z'\cos\theta$$

$$\Rightarrow \vec{\underset{\sim}{A}}(\vec{r}) \cong \frac{\mu_0}{4\pi} \tilde{I}_0 \frac{e^{ikr}}{r} \int_{-s/2}^{s/2} \left[1 + \left(-ik - \frac{1}{r}\right) z'\cos\theta\right] dz' \, \hat{z}$$

$$\cong \frac{\mu_0}{4\pi} \tilde{I}_0 \frac{e^{ikr}}{r} \left(s + \left(-ik - \frac{1}{r}\right)\frac{s^2}{4}\cos\theta + \cdots\right) \hat{z}$$

Substitute in for $\tilde{I}_0 = -i\omega q_0 = -ick\, q_0$

$$\Rightarrow \vec{\underset{\sim}{A}}(\vec{r}) \cong -\frac{i\mu_0 c}{4\pi} q_0 \frac{e^{ikr}}{r} \left(\underbrace{ks}_{\substack{\text{first order} \\ \text{in small parameters}}} + \underbrace{\left(-i(ks)^2 + ks\left(\frac{s}{r}\right)\right)\cos\theta}_{\substack{\text{second order} \\ \text{in small parameters}}} + \cdots\right)$$

To first order

$$\Rightarrow \boxed{\vec{\underset{\sim}{A}}(\vec{r}) = -\frac{i}{4\pi\epsilon_0} \frac{k}{c} \vec{p}_0 \frac{e^{ikr}}{r}}$$

where $\vec{p}_0 = q s_0 \hat{z}$

$$\underset{\sim}{V}(\vec{r}) = \frac{q_0}{4\pi\epsilon_0}\left\{\frac{e^{ik r_+}}{r_+} - \frac{e^{ikr_-}}{r_-}\right\}$$

$$r_\pm = \sqrt{r^2 \mp 2r\frac{s}{2}\cos\theta + \frac{s^2}{4}} \approx r\left(1 \mp \frac{s}{2r}\cos\theta\right)$$

(Approx: # 1)

$$\Rightarrow \underset{\sim}{V}(\vec{r}) \cong \frac{q_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left(\frac{e^{-ik\frac{s}{2}\cos\theta}}{1-\frac{s}{2r}\cos\theta} - \frac{e^{+ik\frac{s}{2}\cos\theta}}{1+\frac{s}{2r}\cos\theta}\right)$$

$$\cong \frac{q_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\frac{1-ik\frac{s}{2}\cos\theta}{1-\frac{s}{2r}\cos\theta} - \frac{1+ik\frac{s}{2}\cos\theta}{1+\frac{s}{2r}\cos\theta}\right]$$

(Approx # 2)

$$\cong \frac{q_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big[\left(1-ik\frac{s}{2}\cos\theta\right)\left(1+\frac{s}{2r}\cos\theta\right) - \left(1+ik\frac{s}{2}\cos\theta\right)\left(1-\frac{s}{2r}\cos\theta\right)\Big]$$

(keeping only first order)

$$\cong \frac{q_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big[\left(1-ik\frac{s}{2}\cos\theta + \frac{s}{2r}\cos\theta\right) - \left(1+ik\frac{s}{2}\cos\theta - \frac{s}{2r}\cos\theta\right)\Big]$$

$$\boxed{\underset{\sim}{V}(\vec{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{p_0\cos\theta\, e^{ikr}}{r^2} - i\frac{1}{4\pi\epsilon_0}kp_0\cos\theta\frac{e^{ikr}}{r}}$$

Can check $\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0}$

Check: Static limit $\omega \to 0$ $\quad k \to 0$

$$V(\vec{r}) \to \frac{1}{4\pi\epsilon_0}\frac{p_0\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \checkmark$$

$$\vec{A}(\vec{r}) = 0 \quad \checkmark$$

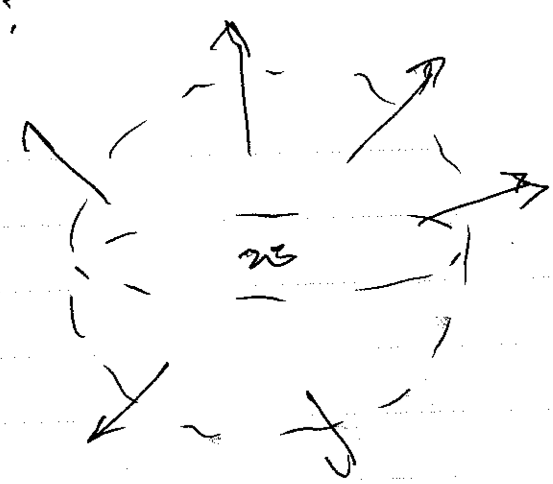
We seek the fields radiated by the oscillating dipole

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \Rightarrow \underset{\sim}{\vec{E}} = i\omega\underset{\sim}{\vec{A}} - \vec{\nabla}\underset{\sim}{V}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \underset{\sim}{\vec{B}} = \vec{\nabla}\times\underset{\sim}{\vec{A}}$$

Aside: Radiated fields from a localized source fall off as $\frac{1}{r}$

Why?



Flux of energy radiated through sphere radius r

$$\oint \vec{S}\cdot d\vec{a} \sim S_{(r)} 4\pi r^2$$

$$\propto (E_{rad}^2)\, r^2$$

Must be independent of r

$$\Rightarrow E_{radiation} \sim \frac{1}{r} \quad \text{(same for B)}$$

for localized source

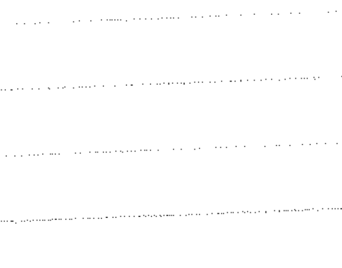
Define the "<u>radiation zone</u>" $r \gg \lambda \gg s$

Look at field @ distances r big compared to λ ("far field"). By assumption $\lambda \gg s$ (non-relativistic)

In "radiation zone", $r \gg \lambda$

$$\boxed{\begin{aligned} \tilde{V}_{rad} &= -i\frac{k p_0 \cos\theta}{4\pi\epsilon_0}\frac{e^{ikr}}{r} \\ \tilde{A}_{rad} &= \frac{-i\frac{k}{c} p_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\,\hat{z} \end{aligned}}$$

$$\Rightarrow \tilde{A}_{rad} = -i\frac{\frac{k}{c}p_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big(\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{\text{expressing } \hat{z} \text{ in spherical coords}}\Big)$$



$$\vec{\nabla}\underset{\sim}{V}_{rad} = \frac{\partial \underset{\sim}{V}_{rad}}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \underset{\sim}{V}_{rad}}{\partial \theta}\hat{\theta}$$

$$\Longrightarrow \frac{k^2 p_0 \cos\theta}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\hat{r} \quad \text{(in radiation zone)}$$

$$i\omega\underset{\sim}{\vec{A}}_{rad} = -\frac{k^2 p_0}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$\vec{\nabla}\times\underset{\sim}{A}_{rad} = \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{A}_\theta) - \frac{1}{r}\frac{\partial \tilde{A}_r}{\partial \theta}\right]\hat{\phi}$$

$$\Longrightarrow -\frac{1}{4\pi\epsilon_0}\frac{k^2}{c}p_0\sin\theta\,\frac{e^{ikr}}{r}\hat{\phi}$$

$$\therefore \boxed{\begin{aligned} \underset{\sim}{\vec{E}}_{rad} &= \frac{1}{4\pi\epsilon_0}k^2 p_0 \sin\theta\,\frac{e^{ikr}}{r}\hat{\theta} \\ \underset{\sim}{\vec{B}}_{rad} &= \frac{1}{4\pi\epsilon_0}\frac{-k^2}{c}p_0\sin\theta\,\frac{e^{ikr}}{r}\hat{\phi} \\ \vec{k} &= \frac{\omega}{c}\hat{r} \end{aligned}}$$

Real Field

$$\boxed{\begin{aligned} \vec{E}_{rad}(\vec{r},t) &= \frac{-k^2 p_0 \sin\theta}{4\pi\epsilon_0}\frac{\cos(kr-\omega t)}{r}\hat{\theta} \\ \vec{B}_{rad}(\vec{r},t) &= \frac{-\frac{k^2}{c}p_0\sin\theta}{4\pi\epsilon_0}\frac{\cos(kr-\omega t)}{r}\hat{\phi} \end{aligned}}$$

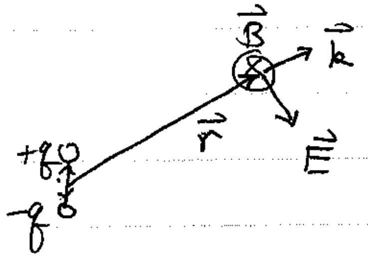
General Properties

- Spherical waves: Surfaces of constant phase are spheres. Wave vector $\perp$ to wave front $\vec{k} = \frac{\omega}{c}\hat{r}$

- Amplitude of radiation is anisotropic – fields vary as a function of θ; $E, B \propto \sin\theta$. This must be true since $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

  in vacuum

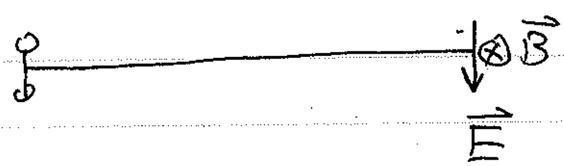
- $\vec{E} \perp \vec{B}$, $\vec{E}, \vec{B} \perp \vec{k}$ $\quad |\vec{B}| \propto \frac{|\vec{E}|}{c}$



$$\vec{E} \times \vec{B} \propto \vec{k}$$

Along x-axis $\theta = \frac{\pi}{2}$, $\phi = 0$

$r = x \qquad \hat{\theta} = -\hat{z}, \quad \hat{\phi} = +\hat{y}$



$$\vec{E} = -E_0(x)\cos(kx - \omega t)\,\hat{z} \qquad \left(E_0(x) = \frac{-k^2 p_0}{4\pi\epsilon_0 x}\right)$$

$$\vec{B} = \frac{E_0(x)}{c}\cos(kx - \omega t)\,\hat{y}$$

Energy radiated

Intensity of radiated field:

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}(\vec{E}^*_{rad} \times \vec{B}_{rad}) = \frac{1}{32\pi^2 \epsilon_0} \, c k^4 \frac{p_0 \sin^2\theta}{r^2} \, \hat{r}$$

Angular distribution of radiated power

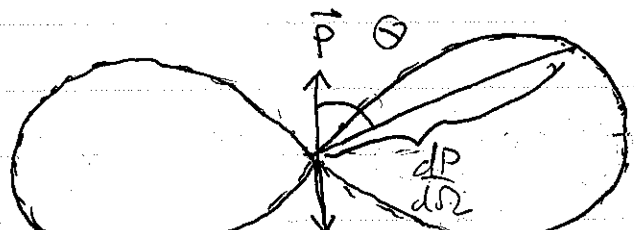
Power radiated into patch on sphere @ θ, ϕ

$$dP = \langle \vec{S} \rangle \cdot \hat{r}\, da$$

$\Rightarrow$ Differential power/solid angle

$$\frac{dP}{d\Omega} = \langle \vec{S} \rangle \cdot \hat{r} r^2 = \left(\frac{c k^4 p_0^2}{32\pi^2 \epsilon_0} \right) \sin^2\theta$$

Polar Plot



"Dipole Radiation Pattern"

The length of the "radius" is proportional to the power radiated at that angle.

(here, independent of ϕ)

The total power radiated into all directions

$$\text{Power} = \int \frac{dP}{d\Omega}\, d\Omega = \frac{c k^4 p_0^2}{32\pi^2 \epsilon_0} \int \sin^2\theta \, d\Omega$$

Aside: Integration over the sphere

$$\int \sin^2\theta \, d\Omega = \int (1-\cos^2\theta) \int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{\int_{-1}^{1} d(\cos\theta)}$$

$$= 2\pi \int_{-1}^{1} (1-\mu^2)\, d\mu \quad (\text{where } \mu = \cos\theta)$$

$$= 2\pi \left(\mu - \tfrac{1}{3}\mu^3\right)\Big|_{-1}^{1} = \frac{8\pi}{3}$$

$$\Rightarrow \boxed{\text{Power}_{\text{radiated}} = \frac{1}{4\pi\epsilon_0} \left(\frac{c k^4 p_0^2}{3} \right)}$$

This is a form of Larmor's formula

- Power radiated ~ square of dipole amplitude
- Power radiated ~ ω^4

  High frequency radiates much more rapidly

Time-domain Results

For the oscillating dipole, we can go back to the time domain using $-i\omega \Leftrightarrow \frac{\partial}{\partial t}$

Here $\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0}\left(-\frac{\omega^2}{c^2} p_0 \frac{\cos(\omega(t-\frac{r}{c}))}{r} \sin\theta\,\hat{\theta}\right)$

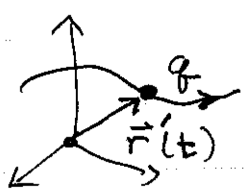
$$\Rightarrow \vec{E}_{rad}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\ddot{p}(t-\frac{r}{c})}{c^2 r}\sin\theta\,\hat{\theta} = \frac{1}{4\pi\epsilon_0}\frac{\ddot{\vec{p}}_\perp(t-\frac{r}{c})}{r c^2}$$

where $p(t) = p_0 \cos\omega t$ — transverse to $\hat{k} = \hat{r}$

radiated field $\propto$ second time derivative of dipole

This generalizes (as we will see) to non-harmonic oscillating dipole:

Consider point charge on a trajectory



Dipole moment relative to origin: $\vec{p}(t) = q\,\vec{r}'(t)$

$$\Rightarrow \vec{E}_{rad}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\ddot{\vec{p}}_\perp(t-\frac{r}{c})}{c^2 r} = \frac{q}{4\pi\epsilon_0}\frac{\ddot{\vec{r}}'_\perp(t-\frac{r}{c})}{c^2 r}$$

$$\boxed{\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2 r}\vec{a}_\perp(t-\frac{r}{c})}$$

radiated field $\propto$ <u>acceleration</u> of charge