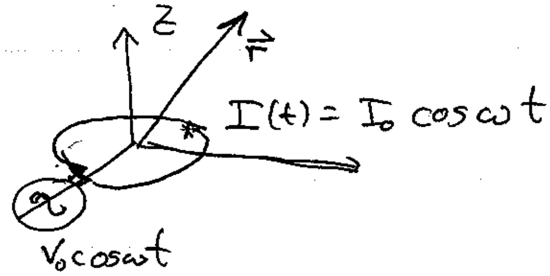
Physics 406

Lecture 20: Dipole Radiation (II)

Magnetic Dipole Radiation

Consider a circular loop of wire ~~driven~~ driven by an AC source



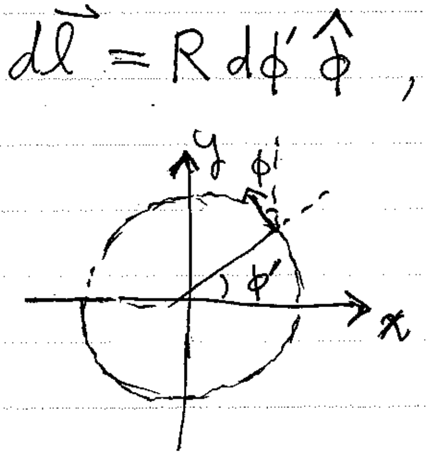
There is a time dependent magnetic dipole moment

$$\vec{m}(t) = m_0 \cos\omega t \qquad \text{where } m_0 = I_0(\pi R^2)$$

Vector potential: $\vec{A}(\vec{r},t) = \mathrm{Re}\left(\underset{\sim}{\vec{A}}(\vec{r})\, e^{-i\omega t}\right)$

$$\underset{\sim}{\vec{A}}(\vec{r}) = \frac{\mu_0}{4\pi}\oint d\vec{l}\; I_0\, \frac{e^{ik\mathscr{r}}}{\mathscr{r}} \qquad \left(\begin{array}{c}\text{From general form}\\ \text{in Lecture 18}\end{array}\right)$$

$$d\vec{l} = R\,d\phi'\,\hat{\phi}, \qquad \hat{\phi} = \cos\phi'\,\hat{y} - \sin\phi'\,\hat{x}$$



Aside: $\mathcal{R} = |\vec{r} - \vec{r}'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}$

On ring: $\vec{r}' = R(\cos\phi'\hat{x} + \sin\phi'\hat{y})$

$\vec{r} = r(\cos\theta\,\hat{z} + \sin\theta(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}))$

$$\Rightarrow \mathcal{R} = \sqrt{r^2 - 2rR\sin\theta\cos(\phi-\phi') + R^2}$$

Multipole Approximation $r >> R$

$\Rightarrow$ To first order in $\frac{R}{r}$

$$\mathcal{R} \approx r\left(1 - \frac{R}{r}\sin\theta\cos(\phi-\phi')\right)$$

Or $\frac{1}{\mathcal{R}} \approx \frac{1}{r}\left(1 + \frac{R}{r}\sin\theta\cos(\phi-\phi')\right)$

Non-Relativistic approximation $\omega R << c \Rightarrow kR << 1$

$$\Rightarrow e^{ik\mathcal{R}} \approx e^{ikr}\, e^{-ikR\sin\theta\cos(\phi-\phi')}$$

$$\approx e^{ikr}\left(1 - ikR\sin\theta\cos(\phi-\phi')\right)$$

$$\Rightarrow \underset{\sim}{\vec{A}}(\vec{r}) \cong \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\oint d\vec{l}\left(1 - ikR\sin\theta\cos(\phi-\phi')\right) \times$$

$$\left(1 + \frac{R}{r}\sin\theta\cos(\phi-\phi')\right)$$

$\Rightarrow$ To first order in $\frac{R}{r}$ and kR

$$\underset{\sim}{\vec{A}}(\vec{r}) \cong \frac{\mu_0}{4\pi} I \frac{e^{ikr}}{r} \oint d\vec{l}' \left(1 + R\sin\theta\cos(\phi-\phi') \left(\frac{1}{r} - ik\right)\right)$$

Aside: $\oint d\vec{l} = 0$

$$\oint d\vec{l}\; R\sin\theta\cos(\phi-\phi') = R^2\sin\theta \int_0^{2\pi} \cos(\phi-\phi')\,\hat{\phi}\,d\phi'$$

$$= R^2\sin\theta \int_0^{2\pi} (\cos\phi\cos\phi' + \sin\phi\sin\phi')(\cos\phi'\hat{y} - \sin\phi'\hat{x})\,d\phi'$$

$$= \pi R^2 \sin\theta\,(\cos\phi\,\hat{y} - \sin\phi\,\hat{x})$$

$$= \pi R^2 \sin\theta\,\hat{\phi}$$

$$\therefore \underset{\sim}{\vec{A}}(\vec{r}) \cong \frac{\mu_0}{4\pi} m_0 \sin\theta\,\hat{\phi}\left(\frac{e^{ikr}}{r^2} \underbrace{- ik\frac{e^{ikr}}{r}}_{\substack{\uparrow \\ \text{radiation component}}}\right)$$

$$\Rightarrow \boxed{\underset{\sim}{\vec{A}}_{rad}(\vec{r}) = \frac{\mu_0}{4\pi}\left(-ikm_0\sin\theta\,\frac{e^{ikr}}{r}\right)\hat{\phi}}$$

Here $V = 0$ (no charge)

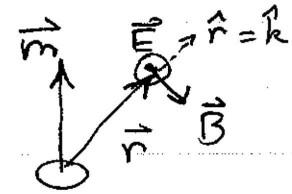
$$\Rightarrow \underset{\sim}{\vec{E}}_{rad} = -\frac{\partial \vec{A}_{rad}}{\partial t} = -i\omega \underset{\sim}{\vec{A}}_{rad} = -ick \underset{\sim}{\vec{A}}_{rad}$$

$$\boxed{\vec{E}_{rad} = \frac{\mu_0}{4\pi}\left(-ck^2 m_0 \sin\theta \frac{e^{i(kr-\omega t)}}{r}\right)\hat{\phi}}$$

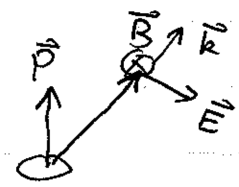
$$\vec{B}_{rad} = \vec{\nabla}\times\vec{A}_{rad} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi(r,\theta)\right)\hat{r}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi(r,\theta)\right)\hat{r}$$

Keeping only terms that go as $\frac{1}{r}$

$$\boxed{\vec{B}_{rad} = \frac{\mu_0}{4\pi}\left(-k^2 m_0 \sin\theta \frac{e^{i(kr-\omega t)}}{r}\right)\hat{\theta}}$$



magnetic dipol



Electric dipole

Intensity of radiation

$$\langle \vec{S} \rangle = \frac{\mu_0^2 c}{32\pi^2} k^4 \frac{m_0^2 \sin^2\theta}{r^2}$$

Same angular distribution

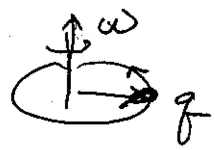
Total radiated P

$$\boxed{P_{mag\text{-}dipole} = \int \langle \vec{S} \rangle \cdot \hat{r}\, r^2 d\Omega = \frac{1}{4\pi\epsilon_0}\left(\frac{k^4 m_0^2}{3c}\right)}$$

Comparing Total Power radiated in magnetic dipole vs. electric dipole

$$\boxed{\frac{P_{\text{mag-dipole}}}{P_{\text{elec-dipole}}} = \left(\frac{m_0/c}{p_0}\right)^2}$$

Example: Charge going in circle



$$p \sim qR$$
$$m \sim \omega q R^2$$

$$\Rightarrow \frac{P_{\text{mag-dipole}}}{P_{\text{elec-dipole}}} \sim \left(\frac{\omega}{c}R\right)^2 \sim (kR)^2 \ll 1$$

$\Rightarrow$ In non-relativistic motion, Electric-dipole dominates over magnetic-dipole by ratio $\left(\frac{v}{c}\right)^2$

Magnetic dipole only important if electric dipole <u>vanishes</u>.

Beyond harmonic time-dependence

Let us return to general expressions for the potentials:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}', t_{ret})}{\eta}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}', t_{ret})}{\eta}$$

where $\quad t_{ret} = t - \frac{\eta}{c} \quad , \quad \eta = |\vec{r}-\vec{r}'|$

Approximation #1: Multipole expansion $r' \ll r \quad \forall r'$ in source

$$\Rightarrow \quad \eta = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2} \cong r\left(1 - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right) = r\left(1 - \hat{r}\cdot\frac{\vec{r}'}{r}\right)$$

$$\frac{1}{\eta} \cong r\left(1 + \frac{\vec{r}\cdot\hat{r}'}{r}\right)$$

$$\rho(\vec{r}', t_{ret}) \approx \rho(\vec{r}', \underbrace{t - \tfrac{r}{c}}_{t_0 \text{ (retarded time from origin)}} + \tfrac{\hat{r}\cdot\vec{r}'}{c})$$

Taylor expansion $\cong \rho(\vec{r}', t_0) + \dot{\rho}(\vec{r}', t_0)\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) + \ldots$

Higher order terms: $\frac{1}{2}\ddot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^2 , \quad \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^3 , \ldots$

Require $\frac{\dot{\rho}(r)}{\rho(r)}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) \ll 1$

Approximation #2 $\quad |\vec{r}'| \ll c\,T \quad$ characteristic time of change of source

$\Rightarrow$ To first order in $\frac{r'}{r}$ and $\frac{r'}{cT}$

$$V(\vec{r},t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\int d^3r'\, \rho(\vec{r}',t_0)}{r} + \frac{\hat{r}\cdot\int \vec{r}'\, \rho(\vec{r}',t_0)\, d^3r'}{r^2} + \frac{\hat{r}\cdot\frac{d}{dt}\int \vec{r}'\rho(\vec{r}';t_0)\, d^3r'}{cr} + \dots\right]$$

$$\Rightarrow \boxed{V(\vec{r},t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{q_0}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{cr} + \dots\right]}$$

where $q_0 = \int d^3r\, \rho(\vec{r}';t_0)$ = total charge (monopole moment)

$\vec{p}(t_0) = \vec{p}\Big|_{t=t-\frac{r}{c}} = \int \vec{r}'\rho(\vec{r}';t_0)$ = electric dipole moment @ time $t-\frac{r}{c}$

This looks like the familiar multipole expansion of the scalar potential with the addition of a new term $\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{cr}$

The vector potential

Aside: Consider $\frac{d\vec{p}(t_0)}{dt} = \frac{d}{dt}\int d^3r'\, \vec{r}'\rho(\vec{r},t_0)$

$$= \int d^3r'\, \vec{r}'\, \frac{\partial\rho(\vec{r}',t_0)}{\partial t} = \int d^3r'\, \vec{r}'\left(-\vec{\nabla}'\cdot\vec{J}(\vec{r}',t_0)\right)$$

integration by parts $\rightarrow$
$$= \int d^3r' \left(\frac{\vec{\nabla}'\cdot\vec{r}'}{3}\right)\vec{J}(\vec{r}',t_0) = \int d^3r'\, \vec{J}(\vec{r}',t_0)$$

$\Rightarrow$ Lemma: $\int d^3r' \; \vec{J}(\vec{r}', t_0) = \frac{d}{dt}\vec{p}(t_0)$

Thus to first order in small parameters

$$\vec{A}(\vec{r}, t) \cong \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \; \vec{J}(\vec{r}', t_0)$$

$$= \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}(t_0)}{r} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{\dot{\vec{p}}(t_0)}{c^2 r}}$$

Fields
$$\begin{cases} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

$$-\vec{\nabla} V = \underbrace{\frac{1}{4\pi\epsilon_0} \frac{q_0}{r^2} \hat{r}}_{\text{quasi-static monopole field}} + \underbrace{\frac{1}{4\pi\epsilon_0} \left(\frac{\vec{p}_0 + 3(\vec{p}_0 \cdot \hat{r})\hat{r}}{r^3} \right)}_{\text{quasi static dipole field}}$$

$$\frac{-1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\hat{r} \cdot \dot{\vec{p}}(t - \frac{r}{c})}{cr} \right) \Longleftarrow \text{New term}$$

$$= \frac{1}{4\pi\epsilon_0} \left(-\frac{\dot{\vec{p}}}{cr^2} + \frac{2(\hat{r} \cdot \dot{\vec{p}})\hat{r}}{cr^2} + \frac{\hat{r}(\hat{r} \cdot \ddot{\vec{p}})}{c^2 r} \right)$$

$$-\frac{\partial \vec{A}}{\partial t} = \frac{-1}{4\pi\epsilon_0} \frac{\ddot{\vec{p}}(t_0)}{c^2 r^2}$$

$$\vec{\nabla} \times \vec{A} = -\frac{\hat{r}}{c} \times \ddot{\vec{p}}(t_0) + \frac{\dot{\vec{p}} \times \hat{r}}{c^2 r}$$

Fields in <u>electric dipole</u> approximation

$$\vec{E}(\vec{r},t) = \frac{q_0}{r^2}\hat{r} + \frac{3\hat{r}(\hat{r}\cdot\vec{p}(t_0)) - \vec{p}(t_0)}{r^3}$$

$$+ \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}(t_0)) - \dot{\vec{p}}(t_0)}{cr^2}$$

$$- \frac{1}{c^2}\left(\frac{\ddot{\vec{p}}(t_0) - \hat{r}(\hat{r}\cdot\ddot{\vec{p}}(t_0))}{r}\right)$$

All times $\frac{1}{4\pi\epsilon_0}$

$$\vec{B}(\vec{r},t) = \frac{\dot{\vec{p}}(t_0)\times\hat{r}}{cr^2} + \frac{\ddot{\vec{p}}(t_0)\times\hat{r}}{c^2 r}$$

Three "zones" of interest:

(1) <u>Near field</u>: $r \ll cT \sim \lambda \Rightarrow$ Quasi-Static

$$\vec{E}(\vec{r},t) \cong \left(\frac{q_0}{r^2}\hat{r} + \frac{3(\hat{r}\cdot\vec{p})\hat{r} - \vec{p}}{r^3}\right)\frac{1}{4\pi\epsilon_0}$$

$$\vec{B}(\vec{r},t) \cong 0$$

(2) <u>Induction zone</u> $r \sim cT \sim \lambda \Rightarrow$ Faraday law but no displacement current

$$\vec{B}(\vec{r},t) \cong \frac{\dot{\vec{p}}(t_0)\times\hat{r}}{cr^2} \Leftarrow (\vec{\nabla}\times\vec{B} = \mu_0\vec{J})$$

$$\vec{E}(\vec{r},t) \cong \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}) - \dot{\vec{p}}}{cr^2}$$

(3) Far field (radiation zone) $r \gg \lambda$

$$\vec{E}(\vec{r},t) \cong \frac{-1}{4\pi\epsilon_0}\left(\frac{\ddot{\vec{p}}(t_0) - \hat{r}\,(\hat{r}\cdot\ddot{\vec{p}}(t_0))}{c^2 r}\right)$$

$$\Rightarrow \boxed{\begin{aligned} \vec{E}_{rad}(\vec{r},t) &= +\frac{1}{4\pi\epsilon_0}\left(\frac{-\ddot{\vec{p}}_{\perp}(t_0)}{c^2 r}\right) \\ \vec{B}_{rad}(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\left(\frac{\ddot{\vec{p}}(t_0)\times\hat{r}}{c^2 r}\right) \end{aligned}}$$

where $\vec{p}_{\perp} = \vec{p} - \vec{p}_{\parallel}$ $\qquad \vec{p}_{\parallel} = (\vec{p}\cdot\hat{r})\,\hat{r}$