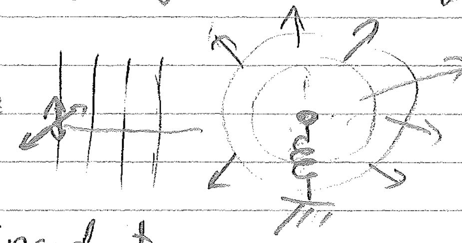
Physics 406: E&M II

Lecture 22: Scattering of Electromagnetic Waves II

Scattering by a Lorentz Oscillator (charge on spring)

Unpolariz



scattered (re-radiated) wave

incident wave

The induced dipole moment $\underset{\sim}{\vec{p}} = \alpha \underset{\sim}{\vec{E}}_{inc}$ ← weak field

$$\tilde{\alpha} = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma} \quad : \text{Polarizability}$$

The differential scattering cross-section (incident wave unpolarized)

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{k^4 |\alpha|^2}{(4\pi\epsilon_0)^2} \left(\frac{1+\cos^2\theta}{2}\right)$$

- Two important limits
  - $\omega \gg \omega_0, \Gamma \Rightarrow \tilde{\alpha} \approx \frac{-e^2}{m\omega^2}$

Approx Free electron ⇒ Thompson scattering

$$\Rightarrow \frac{d\sigma_{Th}}{d\Omega}(\theta) \approx k^4 \left(\frac{e^2/4\pi\epsilon_0}{m\omega^2}\right)^2 \left(\frac{1+\cos^2\theta}{2}\right)$$

$$= r_c^2 \left(\frac{1+\cos^2\theta}{2}\right)$$

where $\boxed{r_c = \frac{e^2/4\pi\epsilon_0}{mc^2} \equiv \text{Classical electron radius}}$

radius at which electronic potential of sphere of charge = rest mass energy

$$\boxed{r_c = 2.8 \times 10^{-15}\ m}$$

Quantum theory - no evidence of any finite radius $\Rightarrow$ point particle ; maybe string ?

Total Thompson Scattering Cross section

$$\sigma_{Thomp} = \int \frac{d\sigma_{Th}}{d\Omega}(\theta)\, d\Omega = \frac{r_c^2}{2} \underbrace{\int (1+\cos^2\theta)\, d\Omega}_{\frac{16\pi}{3}}$$

$$\Rightarrow \boxed{\begin{aligned}\sigma_{Thomp} &= \frac{8\pi}{3} r_c^2 = 0.665 \times 10^{-24}\ cm^2 \\ &= 0.665\ \text{barns}\end{aligned}}$$

Note: Thompson formula is valid $\lambda \gg \frac{h}{mc} = \lambda_{Compton}$

Otherwise "particle nature" of photon matters

$\Rightarrow$ Compton scattering

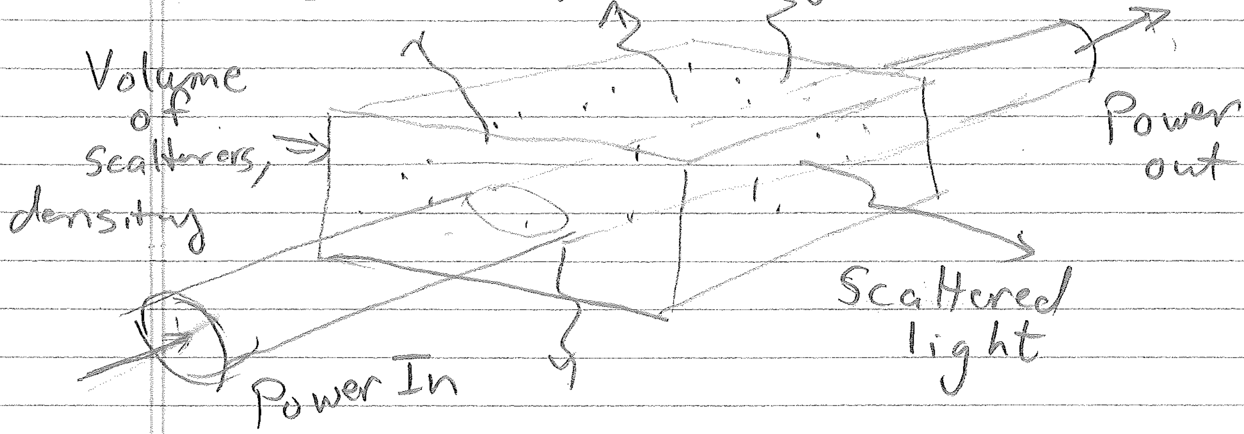
- Rayleigh Scattering $\quad \omega_0 \gg \omega, \Gamma$

$$\Rightarrow \alpha_{Ray} \cong \frac{e^2}{m\omega_0^2} \ (\text{constant}) = \frac{\omega^2}{\omega_0^2} \alpha_{Thomp}$$

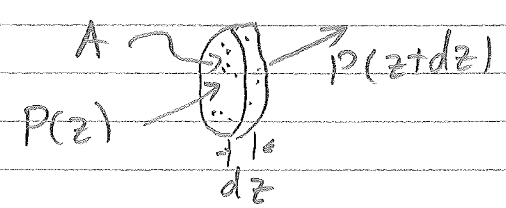
$$\Rightarrow \boxed{\sigma_{Rayleigh} = \frac{8\pi}{3} r_c^2 \left(\frac{\omega}{\omega_0}\right)^4 = \sigma_{Thomp} \times \left(\frac{\omega}{\omega_0}\right)^4}$$

$\Rightarrow$ For Rayleigh scattering, the cross section is a very strong function of frequency $\sim \omega^4$: Reason; in this regime the dipole amplitude independ of frequency, but acceleration $\propto \omega^2$

Attenuation by Scattering



Consider a differential slab



\# of scatterers in slab

$$N = nA\,dz$$

Differential change in power transmitted through slab

$$dP = P(z+dz) - P(z) = -dP_{scat} = -N\underbrace{\sigma I(z)}_{\text{scattered power/atom}}$$

$$\Rightarrow A\underbrace{dI}_{\text{differential change in intensity}} = -nA\sigma\, dz\, I(z)$$

$$\Rightarrow \boxed{\frac{dI}{dt} = -n\sigma I}$$

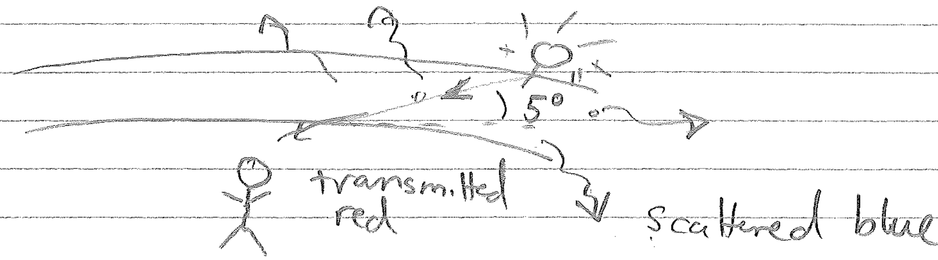
$$\Rightarrow \boxed{I(z) = e^{-n\sigma z} I(0)}$$

Exponential decay = "Beer's law"

$$\boxed{\text{Decay Length} = \frac{1}{n\sigma} = \text{"Mean free path" for scattering}}$$

Why the Sun is red at Sunset

Sunsets are red because the optical depth of the atmosphere is thicker at sun set

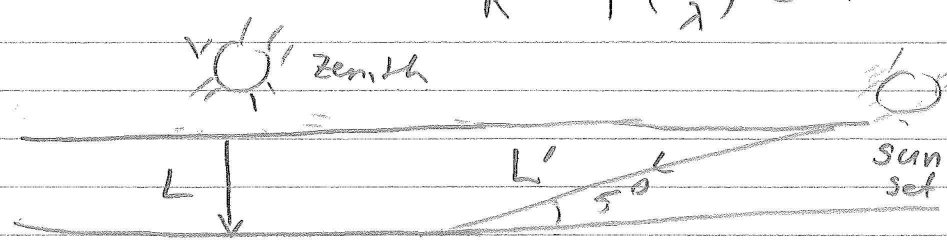


The light we see, looking at the sun, has had much of blue frequencies removed.

The fraction of Power Scattered

$$\text{frac-scat} = \frac{I_{inc} - I_{trans}}{I_{inc}} = 1 - e^{-OD}$$

- For the atmosphere @ zenith: $nL \cong 1.7 \times 10^{25}\ cm^{-1}$
- For blue light @ $\lambda = 450\,nm$: $\sigma_R = \sigma_T \left(\frac{\lambda_0}{\lambda}\right)^4 = 2.17 \times 10^{-16}\ cm^2$



$$L' \cong L/\cos 5° = 11.5\,L$$

$\Rightarrow$ Frac-scat @ zenith $= 1 - e^{-nL\sigma_R} = 8.6 \times 10^{-3}$

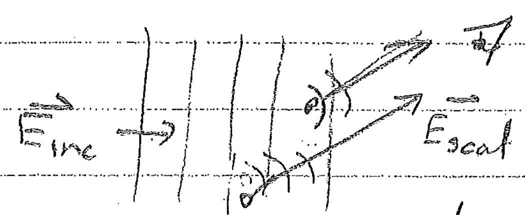
$\Rightarrow$ Sun looks white (not much scattered)

$\Rightarrow$ Frac-scat @ sunset $= 1 - e^{-nL'\sigma_R} \cong 0.1$

$\Rightarrow$ Sun look redish (a decent fraction scattered)

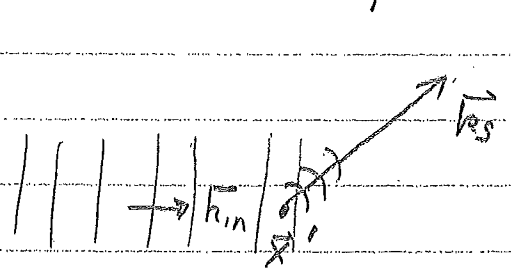
Rayleigh scattering and the "structure factor"

When we derived Beer's Law we implicitly assumed that we could add the scattered power from each radiating dipole. But we should have added the fields not the intensity. How did we get away with this? Should we have gotten away with this?



Different scatterers see different phases of the incident wave. Also, the path length from the scatterer to the observer is different, so accumulated phase is different.

Consider a scatterer @ $\vec{x}'$, observation point $\vec{x}$



Dipole radiation: $\underset{\sim}{\vec{E}}_{scat} = \frac{k^2}{4\pi\epsilon_0} \vec{p}_\perp \frac{e^{ik_s|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$

In the far-field $|\vec{x}| >> |\vec{x}'|$ $\quad |\vec{x}-\vec{x}'| \simeq r - \hat{r}\cdot\vec{x}'$

$$\frac{e^{ik_s|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \simeq \frac{e^{ik_s r}}{r} \; e^{-ik_s \hat{r}\cdot\vec{x}'}$$

Extra phase accumulated

Spherical wave

The induced dipole moment

$$\vec{p}_{i} = \tilde{\alpha}\, \vec{E}_{inc}(\vec{x}') = \tilde{\alpha}\, \vec{E}_0\, \underbrace{e^{i\vec{k}_i \cdot \vec{x}'}}$$

phase of oscillation @ position of dipole

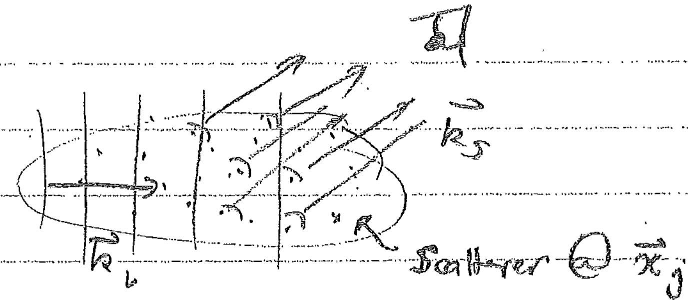
$$\Rightarrow \boxed{\vec{E}_{scat} = \frac{k^2 \tilde{\alpha}}{4\pi\epsilon_0} \vec{E}_{0\perp}\, e^{i(\vec{k}_i - \vec{k}_s)\cdot \vec{x}'}\, \frac{e^{ik_s r}}{r}}$$

Forward scattering : When $\vec{k}_s = \vec{k}_i$

$$\vec{E}_{scat} = k^2 \tilde{\alpha}\, \vec{E}_{0\perp}\, \frac{e^{ik_s r}}{r}$$

independent of $\vec{x}'$ $\Rightarrow$ All scattered waves in the forward direct add in phase

Collection of scatterers



$$\vec{E}_{scat} = \frac{k^2 \tilde{\alpha}}{4\pi\epsilon_0} \vec{E}_{0\perp} \frac{e^{ikr}}{r} \underbrace{\left(\sum_j e^{-i\vec{q}\cdot\vec{x}_j} \right)} \qquad (\vec{q} = \vec{k}_s - \vec{k}_i$$

Phasor addition depending on positions of scatterers and direction of scattered wave relative to incident wave.

We can turn the discrete sum over scatterers into a continuous integral by coarse-graining

$$\boxed{\sum_j e^{-i\vec{q}\cdot\vec{x}_j} = \int d^3x'\, \underset{\substack{\uparrow \\ \text{density of scatterers}}}{n(\vec{x}')}\, e^{-i\vec{q}\cdot\vec{x}'} = \tilde{n}(\vec{q})}$$

$\Rightarrow$ Scattered field proportional to Fourier transform of the density distribution @ $\vec{q} = \vec{k}_s - \vec{k}_i$

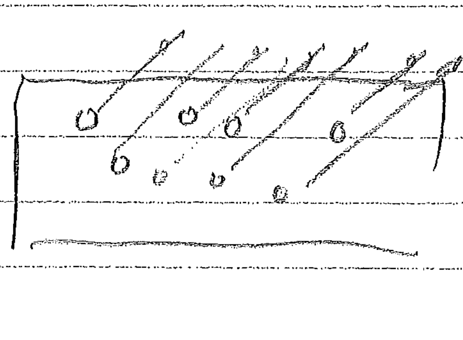
Example: Continuous smooth, uniform dielectric (glass, water)

$$n(\vec{x}) = n_0 \qquad \text{(inside glass)}$$

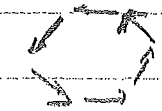
$$\Rightarrow \tilde{n}(\vec{q}) = n_0 \int d^3x'\, e^{-i\vec{q}\cdot\vec{x}'} = n_0\,\delta^{(3)}(\vec{q}) = n_0\,\delta^{(3)}(\vec{k}_s - \vec{k}_i)$$

$\Rightarrow$ $\boxed{\vec{E}_{scat} = 0 \text{ except in forward direction}}$

In all directions other than $\vec{k}_s = \vec{k}_i$ (forward) the scattered waves have every possible phase

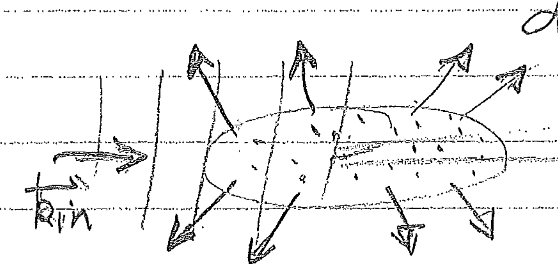


$$\sum_j e^{-i\vec{q}\cdot\vec{x}_j} = 0 \qquad \vec{q} \neq 0$$



Rayleigh

So diffuse scattering is due to density fluctuations



If we write $n(\vec{x}) = \bar{n} + \delta n(\vec{x})$

(mean density) (fluctuation around mean)

$$\Rightarrow \tilde{n}(\vec{q}) = \bar{n}\,\delta^{(3)}(\vec{q}) + \delta\tilde{n}(\vec{q})$$

$\Rightarrow$ Diffuse scattering into 4π due solely to density fluctuations

Structure factor

From our expression for the scattered field

$$I_{scat}(\vec{q}) \propto \left|\sum_j e^{-i\vec{q}\cdot\vec{x}_j}\right|^2$$

$$= \sum_{i,j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)}$$

If positions of scatterers are random then we average over some probability distribution of position

$$\Rightarrow \quad I_{scat}(\vec{q}) \propto \underbrace{\left\langle \sum_{i,j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle}$$

$\equiv S(\vec{q})$: "Structure factor"

(sometimes normalized by N scatterers)

Note: If we break up the sum for $i=j$, $i \neq j$

$$\Rightarrow S(\vec{q}) = \underset{\substack{\uparrow \\ \text{incoherent} \\ \text{addition of scatters}}}{N} + \underbrace{\left\langle \sum_{i\neq j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle}_{\substack{\uparrow \\ \text{coherent contribution} \\ \text{(interference)}}}$$

For $\vec{q}=0$ $\displaystyle\sum_{i\neq j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} = N(N-1)$

$$\Rightarrow \boxed{S(\vec{q}=0) = N^2 \Rightarrow \text{In first Born scattered intensity } \propto N^2 \text{: interference!}}$$

For $\vec{q} \neq 0$, if positions are random

then $\left\langle \displaystyle\sum_{i\neq j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle = 0$

$$\Rightarrow \boxed{S(\vec{q}\neq 0) = N \text{ for random scatterers} \Rightarrow \text{Beer's Law}}$$

Said another way, for random positions, the fluctuation in density $\propto \sqrt{\bar{n}}$

$\bar{n}$ = mean density

Since $\vec{E}_{scat}(\vec{q} \neq 0) \propto$ density fluctuations

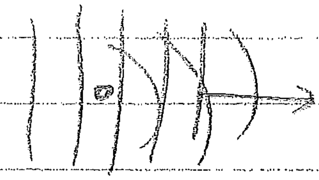
$$\propto \sqrt{\bar{n}}$$

$$I_{scat} \propto |\vec{E}_{scat}|^2 \propto \bar{n} \propto N$$

"incoherent"

<u>Index of refraction</u>

The forward scattered wave can <u>interfer</u> with the incident wave



This interference can phase shift the arrival of a peak or valley of the wave. This changes the velocity of a phase of the wave $\Rightarrow$ <u>index of refraction</u>

The imaginary part of the index corresponds to <u>destructive interference</u> of the forward wave with the incident wave.