Physics 406

Problem Set #1 – Solutions

(1) (a) Show $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}(\vec{r})) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}(\vec{r})) - \nabla^2\vec{F}(\vec{r})$

Let $\vec{G}(\vec{r}) = \vec{\nabla}\times\vec{F} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \hat{x}\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right) + \hat{y}\left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right) + \hat{z}\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)$$

$$= \hat{x}\left(\frac{\partial^2 F_y}{\partial x\partial y} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_z}{\partial x\partial z}\right)$$
$$+ \hat{y}\left(\frac{\partial^2 F_z}{\partial y\partial z} - \frac{\partial^2 F_y}{\partial z^2} - \frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_x}{\partial y\partial x}\right)$$
$$+ \hat{z}\left(\frac{\partial^2 F_x}{\partial z\partial x} - \frac{\partial^2 F_z}{\partial x^2} - \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_y}{\partial z\partial y}\right)$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \hat{x}\frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \hat{y}\frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_z}{\partial z}\right) + \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$
$$- \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_x\hat{x} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)F_y\hat{y} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F_z\hat{z}$$

Now

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$
$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}\right)$$

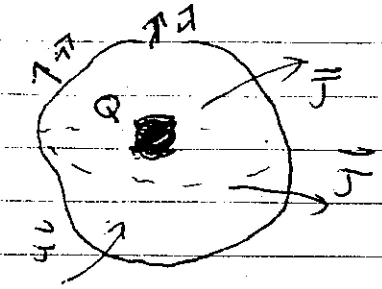
The terms $\frac{\partial^2 F_x}{\partial x^2}$, $\frac{\partial^2 F_y}{\partial y^2}$, $\frac{\partial^2 F_z}{\partial z^2}$ cancel, reproducing the result above.

QED

whew!

1(b) General Conservation Laws:

Consider a closed volume $\bar{V}$



The amount of "Q" inside $\bar{V}$ changes because Q flows in or out of volume or is created/destroyed

$$\frac{dQ}{dt} = \underset{\substack{\uparrow \\ \text{decreases } Q}}{-}(\text{Rate of flow out of } \bar{V}) + (\text{Rate of creation inside } \bar{V})$$

$$= -\int_S \vec{J}\cdot\hat{n}\,da + \int_{\bar{V}} R\,d^3r$$

where S is the surface bounding $\bar{V}$

Now, according to the divergence theorem

$$\int_S \vec{J}\cdot\hat{n}\,da = \int_{\bar{V}} \vec{\nabla}\cdot\vec{J}\,d^3r$$

and if ρ is the local density of Q

$$Q = \int \rho\,d^3r$$

$$\Rightarrow \frac{d}{dt}\int \rho\,d^3r = -\int_{\bar{V}} (\vec{\nabla}\cdot\vec{J})\,d^3r + \int_{\bar{V}} R\,d^3r$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} + R}$$

This is a local conservation law at any point $\vec{r}$

(1c) Complex numbers

(i) $$(-1-i)^2 = 1+2i+(-i)^2 = \boxed{2i = 2e^{i\pi/2}}$$

another way $$(-1-i)^2 = (1+i)^2 = \left(\sqrt{2}\, e^{i\pi/4}\right)^2$$

polar form

$$\Rightarrow (-1-i)^2 = 2e^{i\pi/2} = 2i$$

(ii) $$\frac{1}{\omega^2-\omega_0^2-i\omega\Gamma} * \frac{(\omega^2-\omega_0^2+i\omega\Gamma)}{(\omega^2-\omega_0^2+i\omega\Gamma)}$$ Multiply top and bottom by conjugate

$$\boxed{= \left[\frac{\omega_0^2-\omega_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\right] + i\left[\frac{\omega\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\right]}$$

Real-imaginary form

$$\frac{1}{\omega^2-\omega_0^2-i\omega\Gamma} = \left(\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\; e^{i\theta}\right)^{-1}$$

where $\theta = \tan^{-1}\left(\frac{-\omega\Gamma}{\omega^2-\omega_0^2}\right)$

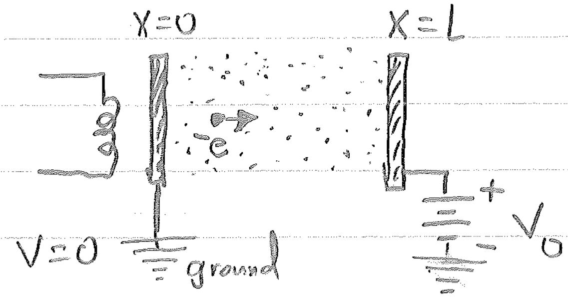
$$\boxed{\frac{1}{\omega^2-\omega_0^2-i\omega\Gamma} = \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}\, e^{-i\theta}}$$

(1c) Continued

(iii) $\sqrt{-i} = \left(e^{-i\pi/2}\right)^{1/2}$ $\boxed{= e^{-i\pi/4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$

(iv) $$\boxed{\frac{e^{-i\omega t}}{\omega^2-\omega_0^2 - i\omega\Gamma} = \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}\, e^{-i(\omega t+\theta)} \quad \text{where } \theta = \tan^{-1}\left(\frac{-\omega\Gamma}{\omega^2-\omega_0^2}\right) = \frac{\cos(\omega t+\theta)}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}} - i\,\frac{\sin(\omega t+\theta)}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}}$$

(2) Space-charge-limited diode



Child-Langmuir Law

$$I = K V_0^{3/2}$$

The electrons boil off of the cathode and are accelerated towards the anode. The total potential seen by an electron is due to the combination of the external potential plus the inter-electron repulsion (space charge).

(a) The local relation between the electrostatic potential $V(x)$ an the charge density $\rho(x) = -e\,n(x)$ (density of electrons) is the Poisson equation

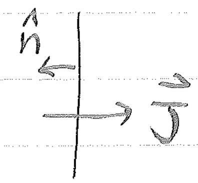
$$\boxed{\nabla^2 V(x) = \frac{d^2V}{dx^2} = -\frac{\rho(x)}{\epsilon_0} = e\,\frac{n(x)}{\epsilon_0}}$$

(b) The <u>key</u> point is that in steady state, the current flowing across the diode is <u>constant</u> (independent of x).

Since $I = |J|A \Rightarrow J$ is independent of x

(we take I to be positive)

Note: The current is flowing from left to right (direction of positive charge)

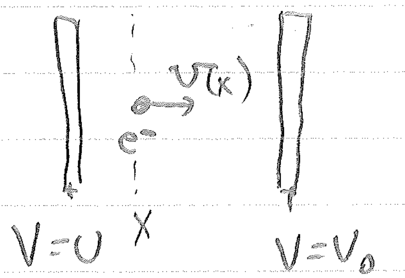


$$\Rightarrow \quad I = -JA$$

Now $\quad J = \rho(x)\, v(x) = -e\, n(x)\, v(x)$

$$\Rightarrow \quad I = e\,n(x)\, v(x)\, A \quad \Rightarrow \quad \boxed{e\,n(x) = \frac{I}{v(x) A}}$$

(c)



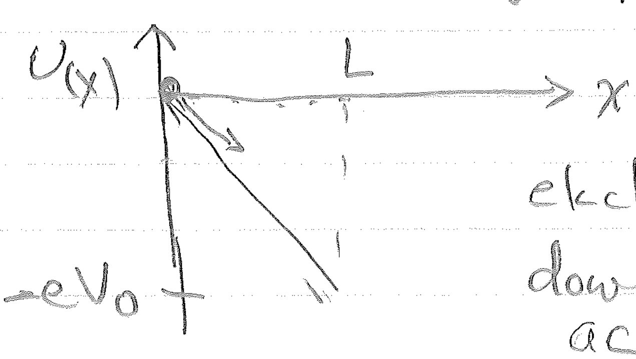
The charge starts at rest and is accelerated to the anode, picking up speed.

Remember, the potential <u>energy</u> of a charge q in an electrostatic potential V is

$$U(x) = q\, V(x)$$

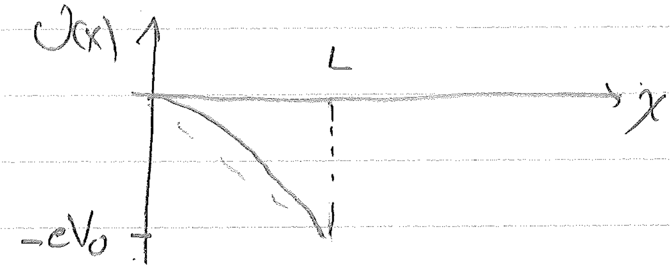
Here $q = -e$ so $U(x) = -e\, V(x)$.

In the absence of space charge $V(x) = \frac{x}{L} V_0$



electron slides down linear hill w/ constant acceleration.

In the presence of space charge we will see $V(x) = C x^{4/3}$



electron rolls down a shallower hill

To derive this, use conservation of energy

$$E = \frac{1}{2} m v^2(x) + U(x) = \frac{1}{2} m v^2(x) - eV(x)$$

@ $x=0$, $v=0$ & $V=0$ $\Rightarrow$ $E=0\ \forall x$ (our choice of ground)

$$\Rightarrow \boxed{v(x) = \sqrt{\frac{2e}{m} V(x)}}$$

(d) Putting (a)-(c) together

$$\frac{d^2V}{dx^2} = \frac{en(x)}{\epsilon_0} = \frac{I}{\epsilon_0 A v(x)} = \sqrt{\frac{m}{2e}} \frac{I}{\epsilon_0 A} V(x)^{-1/2}$$

$$\Rightarrow \boxed{\frac{d^2V}{dx^2} = C V^{-1/2} \quad \text{where} \quad C = \frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2e}}}$$

(e) We claim $V(x) = \left(\frac{9C}{4}\right)^{2/3} x^{4/3}$ is the solution

Check: $\frac{dV}{dx} = \frac{4}{3}\left(\frac{9C}{4}\right)^{2/3} x^{1/3} \Rightarrow \frac{d^2V}{dx^2} = \frac{4}{9}\left(\frac{9C}{4}\right)^{2/3} x^{-2/3}$

$$\Rightarrow \boxed{\frac{d^2V}{dx^2} = C\left(\frac{9C}{4}\right)^{-1/3} x^{-2/3} = C V^{-1/2}} \quad \checkmark$$

Boundary conditions: $V(x=0) = 0$, $V(x=L) = V_0$

$$\Rightarrow V(x=L) = \left(\frac{9C}{4}\right)^{2/3} L^{4/3} = V_0$$

$$\Rightarrow \left(\frac{9}{4}\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2e}}\right)^{2/3} L^{4/3} = V_0$$

Solving for I:

$$\boxed{I = \underbrace{\frac{4\epsilon_0}{9L^2}\left(\frac{2e}{m}\right)^{1/2}}_{K} V_0^{3/2}}$$

The constant K plays the role of conductivity, and depends only on L and fundamental constants.