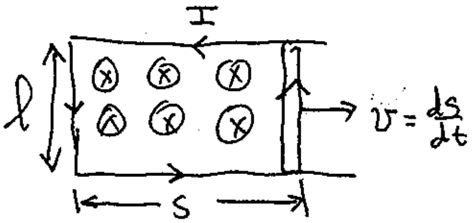
Physics 406: Problem Set #2 Solutions

Problem 1



(a) Faraday's Law: $\underset{\text{emf}}{\mathcal{E}} = -\frac{d\Phi}{dt}$ ← magnetic flux

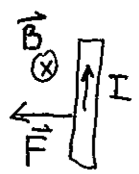
Lenz's Law: In order to counteract the flux increasing into the page, the current will flow to create B-field out of page $\Rightarrow$ $\boxed{I \text{ counter-clockwise}}$

$$I = \frac{\mathcal{E}}{R}, \quad R = \frac{l}{\sigma A} \quad (\text{resistance})$$

$$|\mathcal{E}| = \frac{d}{dt}(Bls) = Blv \quad \left(\text{since } v = \frac{ds}{dt}\right)$$

$$\therefore \quad I = \frac{Blv}{R} = \boxed{B\sigma A v}$$

(b) Magnetic force: Lorentz: $\vec{F} = l\,\vec{I} \times \vec{B}$



$\Rightarrow$ $\boxed{F = BIl = B^2\sigma l A v \text{ (opposing motion of rod)}}$

(c) Newtons Law: $M\dot{v} = \underset{\text{opposes motion}}{-F} = -B^2\sigma l A v$

$$\Rightarrow \dot{v} = -\frac{B^2\sigma l A}{M} v, \quad M = \rho_M A l \quad \left(\text{mass density} \times \text{volume of cylinder}\right)$$

$$\Rightarrow \dot{v} = -\Gamma v \quad \text{where} \quad \Gamma = \frac{B^2\sigma}{\rho_m}$$

Solution $\boxed{v(t) = v_0 e^{-\Gamma t}}$

(d) Given $v(t) = v_0 e^{-\Gamma t}$ At $t \to \infty$ $v \to 0$. Where does the kinetic energy of the rod go?

<u>Answer</u>: Energy is dissipated in the resistive rod via Joule heating.

voltage = EMF

Rate of energy dissipation: $P_{diss}(t) = \mathcal{E}(t)\, I(t)$

$$\therefore\ P_{diss}(t) = \left(B \ell v(t)\right)\left(B \sigma A\, v(t)\right) = B^2 \ell \sigma A\, v^2(t)$$

$$= B^2 \ell \sigma A v_0^2 e^{-2\Gamma t}$$

Total energy dissipated $t=0 \to t=\infty$

$$W_{diss} = \int_0^\infty P_{diss}(t)\, dt = B^2 \ell \sigma A\, v_0^2 \int_0^\infty e^{-2\Gamma t}\, dt$$

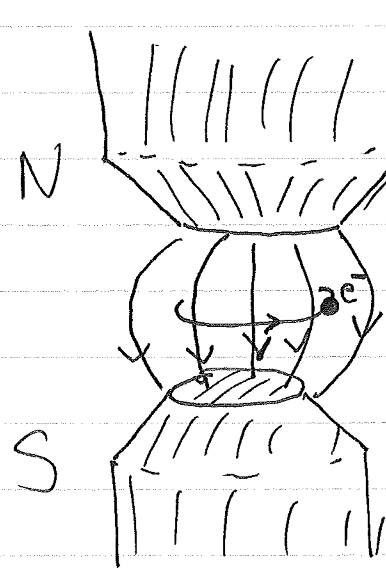
$$= B^2 \ell \sigma A v_0^2 \left(\frac{e^{-2\Gamma t}}{-2\Gamma}\right)_0^\infty = \frac{B^2 \sigma \ell A\, v_0^2}{2\Gamma}$$

Plug in $\Gamma = \frac{B^2 \sigma}{\rho_M}$

$$\Rightarrow W_{diss} = \frac{1}{2}\left(\rho_M \ell A\right) v_0^2 = \frac{1}{2} m v_0^2$$

Problem 2

Griffiths 7.48: The Betatron



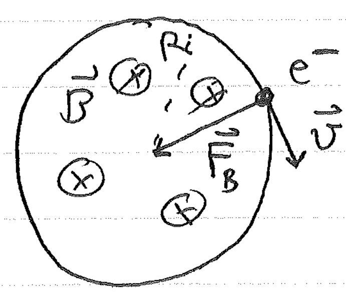
An electron can be accelerated by the electric field induced by a changing magnetic field

(Faraday's Law)

Let $B(R)$ = magnetic field @ the radius of orbit.

The Lorentz force causes the electron to execute cyclotron motion

Top view



$$\vec{F}_B = -e\vec{v} \times \vec{B}(R)$$

$$|\vec{F}_B| = \frac{mv^2}{R} = evB(R)$$

Centripetal force

$$\Rightarrow \quad v = \frac{eB(R)R}{m}$$

Now we let $\vec{B}$ change as a function of time. We seek the conditions such that R is constant

$$\Rightarrow v(t) = \frac{e\,B(R,t)\,R}{m}$$

The acceleration is thus related to the time rate of change of $\vec{B}$ according to

$$\frac{dv}{dt} = \frac{eR}{m}\frac{dB(R,t)}{dt}$$

Now, the tangential acceleration is provided by the curling electric field according to Faraday's Law $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,da$

$$\Rightarrow 2\pi R\; E(R,t) = -\frac{d}{dt}\Phi_B(t)$$

Define the magnetic field averaged over the surface

$$B_{ave}(t) = \frac{1}{\pi R^2}\int \vec{B}\cdot\hat{n}\,da = \frac{1}{\pi R^2}\Phi_B(t)$$

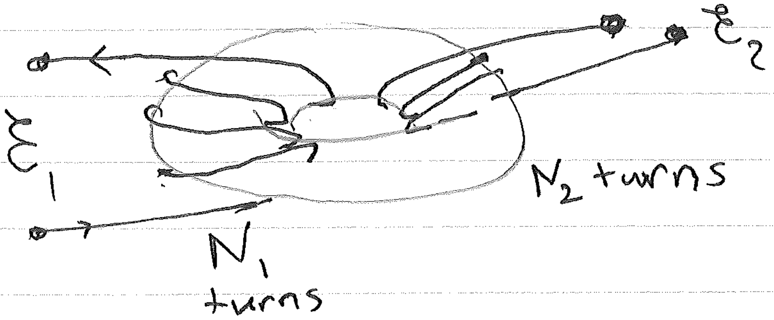
$$\Rightarrow E(R,t) = -\frac{1}{2}R\frac{dB_{ave}}{dt}$$

$\Rightarrow$ Tangential acceleration $\frac{dv}{dt} = \frac{eE}{m} = \frac{eR}{2m}\frac{dB_{ave}}{dt}$

Equating $\Rightarrow$ $\boxed{B(R) = \frac{1}{2}B_{ave}}$ The magnetic field @ R is $\frac{1}{2}$ the average

Problem 3: Transformers

Griffiths 7.53: In a transformer, the <u>same magnetic field lines</u> pass through two coils



Here I draw the transformer as coils wrapped around an iron ~~[illegible]~~ torus.

The key is that

$$\Phi_1 = N_1 \Phi \quad , \quad \Phi_2 = N_2 \Phi$$

where Φ = flux / turn

By Faraday's law, $\mathcal{E}_1 = -N_1 \frac{d\Phi}{dt}$

$$\mathcal{E}_2 = -N_2 \frac{d\Phi}{dt}$$

$$\Rightarrow \frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2} \Rightarrow \boxed{\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{N_1}{N_2}}$$

One can "step-up" or "step-down" the voltage in the secondary loop relative to the first loop.

The transformer is a key component in AC Power lines.

Griffiths 7.54: What about energy conservation?

Though we can change the Voltage, energy is conserved. The current must compensate.

~~Remember~~ Remember, Power = IV

$\Rightarrow$ if energy is conserved if V goes up then I goes down.

This is the main reason the power is delivered on our grid via AC power, so we can easily trade off current for voltage. For long distance power transmission, it is better to have high voltage and low current since the dissipation $= I^2 R$ (want to avoid dissipating electric power to heat by keeping I low) (However, for application in buildings (like the home), we want low voltage to avoid electrocution, Thus, before the power

enters the home, the voltage is stepped down using a transformer. We can do this ~~e~~ easily if the current is oscillating.

The ability to use a transformer is the reason power is AC

(a) The mutual inductance of an ideal transformer is easily related to the self-inductances

In the ideal transformer:

$$\Phi_1 = N_1 \Phi \quad \text{and} \quad \Phi_2 = N_2 \Phi$$

magnetic flux/turn

Now by definition:

self inductance — mutual inductance

$$\Phi_1 = L_1 I_1 + M I_2 = N_1 \Phi$$

$$\Phi_2 = L_2 I_2 + M I_1 = N_2 \Phi$$

Dividing the two equations:

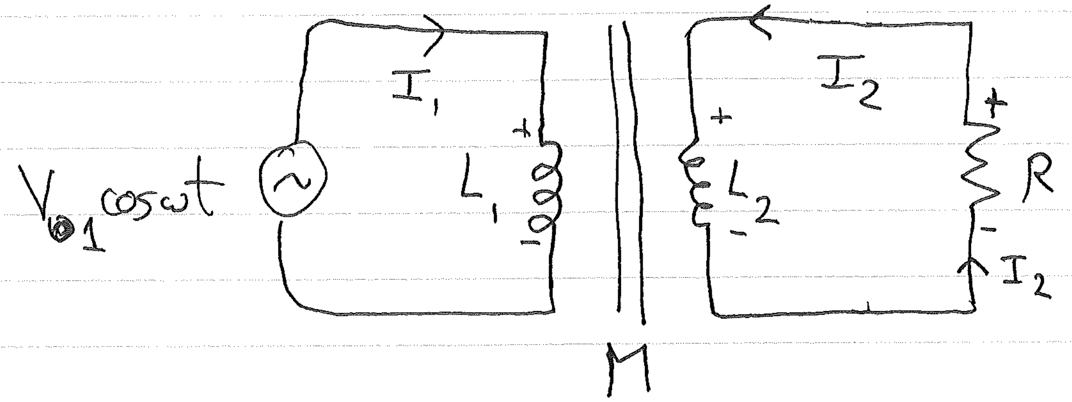
$$\frac{L_1 I_1 + M I_2}{L_2 I_2 + M I_1} = \frac{N_1}{N_2}$$

This is true $\forall I_1, I_2$

For $I_1 = 0$ $\quad \frac{M}{L_2} = \frac{N_1}{N_2}$, $\qquad$ For $I_2 = 0$, $\frac{L_1}{M} = \frac{N_1}{N_2}$

$$\Rightarrow \frac{M}{L_2} = \frac{L_1}{M} \Rightarrow \boxed{M^2 = L_1 L_2}$$

(b) The circuit diagram for a transformer appears as

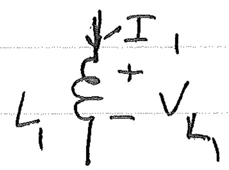


$R \leftrightarrow V_R = -I_2 R$ (by the choice how I defined current)

Using Kirkoff's Law:

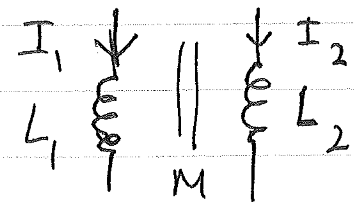
$$V_1 \cos\omega t = V_{L_1}, \qquad V_{L_2} = -I_2 R$$

Recall convention



$$\Rightarrow V_{L_1} = L \frac{dI_1}{dt}$$

In addition there is a contribution from the mutual inductance



$$V_{L_1} = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$V_{L_2} = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} V_1 \cos\omega t &= L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \\ L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} &= -I_2 R \end{aligned}}$$

(c) We can most easily solve this ~~using~~ using complex impedance

$$V_1 \cos\omega t = \mathrm{Re}\,(V_1 e^{-i\omega t})$$

$$I_{1,2}(t) = \mathrm{Re}\,(\tilde{I}_{1,2}\, e^{-i\omega t})$$

$$\Rightarrow \left.\begin{aligned} V_1 &= -i\omega L_1 \tilde{I}_1 - i\omega M \tilde{I}_2 \\ -i\omega L_2 \tilde{I}_2 - i\omega M \tilde{I}_1 &= -\tilde{I}_2 R \end{aligned}\right\} \text{Steady state}$$

$$\Rightarrow \tilde{I}_2 = \left(\frac{-i\omega M}{i\omega L_2 - R}\right)\tilde{I}_1$$

$$\Rightarrow V_1 = \left(-i\omega L_1 - \frac{\omega^2 M^2}{i\omega L_2 - R}\right)\tilde{I}_1$$

$$V_1 = \frac{\omega^2 \cancelto{0}{(M^2 - L_1 L_2)} + i\omega R L_1}{i\omega L_2 - R}\,\tilde{I}_1$$

$$\Rightarrow \boxed{\tilde{I}_1 = \frac{i\omega L_2 - R}{i\omega R L_1}\, V_1}$$

$$\boxed{\tilde{I}_2 = \frac{-M}{R L_1} V_1 = -\sqrt{\frac{L_2}{L_1}} \frac{V_1}{R}}$$

Now, the real currents

$$I_1(t) = \mathrm{Re}\left(\frac{i\omega L_2 - R}{i\omega R L_1} V_1 e^{-i\omega t} \right)$$

$$= \mathrm{Re}\left(\frac{L_2}{L_1} \frac{V_1}{R} e^{-i\omega t} \right) + \mathrm{Re}\left(\frac{V_1}{\omega L_1} i e^{-i\omega t} \right)$$

$$\Rightarrow \boxed{I_1(t) = \left(\frac{L_2}{L_1}\right) \frac{V_1}{R} \cos\omega t + \frac{V_1}{\omega L_1} \sin\omega t}$$

$$\boxed{I_2(t) = \mathrm{Re}\left(\tilde{I}_2 e^{-i\omega t} \right) = -\sqrt{\frac{L_2}{L_1}} \frac{V_1}{R} \cos\omega t}$$

Note: The current in the primary coil has a component in quadrature (going like $\sin\omega t$) with respect to the source

(d) This is a repeat of Griffiths 7.53

(e) The input power (instantaneous)

$$P_{in}(t) = V_1(t)\, I_1(t)$$

$$\boxed{P_{in}(t) = \frac{L_2}{L_1}\frac{V_1^2}{R}\cos^2\omega t + \frac{V_1^2}{\omega L_1}\cos\omega t \sin\omega t}$$

The output power

$$P_{out}(t) = V_2(t)\, I_2(t) = I_2^2(t)\, R$$

$$\boxed{P_{out} = \frac{L_2}{L_1}\frac{V_1^2}{R}\cos^2\omega t}$$

Time averaged: $\overline{\cos^2\omega t} = \frac{1}{2}$ $\qquad \overline{\sin\omega t\cos\omega t} = 0$

$$\Rightarrow \boxed{\overline{P_{in}} = \overline{P_{out}} = \frac{1}{2}\frac{L_2}{L_1}\frac{V_1^2}{R}}$$

As expected, after transients, the power in the primary coil is equal to that in the secondary

<u>Note:</u> You can easilly check that, using the complex amplitudes,

$$\overline{P_{in}} = \frac{1}{2}\,\mathrm{Re}\,(\tilde{I}_1^{*} V_1) = \overline{P_{out}} = \frac{1}{2}\,\mathrm{Re}\,(\hat{I}_2^{*} V_2)$$