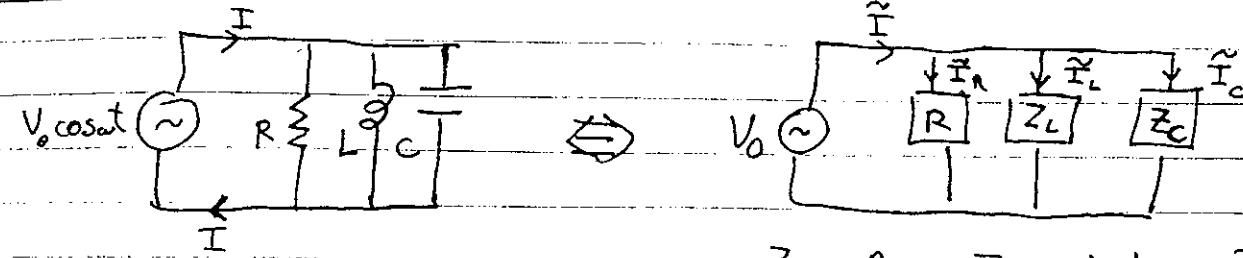
Physics 406

Problem Set #3 Solutions

Problem 1: Parallel R-L-C circuit



$$Z_R = R \quad Z_L = -i\omega L \quad Z_C = \frac{i}{\omega C}$$

Since the elements are in parallel

$$V_R = V_L = V_C = V_0$$

The total current divides between the elements

$$\tilde{I} = \tilde{I}_R + \tilde{I}_L + \tilde{I}_C = \tilde{V}_R / Z_R + \tilde{V}_C / Z_C + \tilde{V}_L / Z_L$$

$$\Rightarrow \tilde{I} = \left(\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} \right) V_0 = \frac{V_0}{Z_{total}}$$

Parallel addition of impedances

$$\frac{1}{Z_{total}} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} - i\omega C + \frac{i}{\omega L}$$

(a) On resonance $\omega = \frac{1}{\sqrt{LC}}$

$$\Rightarrow \frac{1}{Z_{total}} = \frac{1}{R} - i\sqrt{\frac{C}{L}} + i\sqrt{\frac{C}{L}} = \frac{1}{R}$$

$$\Rightarrow \tilde{I} = \frac{V_0}{R} \Rightarrow \boxed{I(t) = \mathrm{Re}(\tilde{I} e^{-i\omega t}) = \frac{V_0}{R} \cos \omega t}$$

On resonance the magnitude of $|Z_L| = |Z_C| = \sqrt{\frac{L}{C}}$ but they are 180° out of phase so they cancel. On resonance a parallel L-C is like an open circuit.

(b) In general

$$I(t) = Re\left(\tilde{I}\, e^{-i\omega t}\right) = Re\left(\frac{V_0}{Z_{total}}\, e^{-i\omega t}\right)$$

$$= \frac{V_0}{|Z_{total}|} \cos\left(\omega t + Arg(Z_{total})\right)$$

$$\frac{1}{|Z_{total}|} = \left|\frac{1}{Z_{total}}\right| = \left|\frac{1}{R} - i\left(\omega C - \frac{1}{\omega L}\right)\right|$$

$$= \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \frac{1}{\omega L}\sqrt{\frac{\omega^2 L^2}{R^2} + (\omega^2 LC - 1)^2}$$

$$= \frac{1}{R}\,\frac{\Gamma}{\omega}\sqrt{\frac{\omega^2}{\Gamma^2} + \left(\frac{\omega^2}{\omega_0^2} - 1\right)}\ , \quad \Gamma = \frac{R}{L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Arg(Z_{total}) = -Arg\left(\frac{1}{Z}\right) = \tan^{-1}\left(\frac{\omega C - \frac{1}{\omega L}}{1/R}\right)$$

$$= \tan^{-1}\left(\frac{\Gamma}{\omega}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right)$$

$$\therefore \boxed{\begin{aligned} I(t) &= \frac{V_0}{R}\,\frac{\Gamma}{\omega}\sqrt{\frac{\omega^2}{\Gamma^2} + \left(\frac{\omega^2}{\omega_0^2} - 1\right)}\ \cos\left(\omega t + Arg(Z_{total})\right) \\ Arg(Z_T) &= \tan^{-1}\left(\frac{\Gamma}{\omega}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right) \end{aligned}}$$

Note: On resonance, $I(t) = \frac{V_0}{R}\cos\omega t$

$Arg(Z_{total}) = 0$

Problem 2 Quality factor

(a) Math Lemma: Given $A(t) = A_0 \cos(\omega t - \phi_A)$
$B(t) = B_0 \sin(\omega t - \phi_B)$

Show $\langle A(t)\, B(t)\rangle \equiv \frac{1}{T}\int_0^T dt\, A(t) B(t) = \frac{1}{2}\,\mathrm{Re}(\tilde{A}^* \tilde{B})$
$= \frac{1}{2}\,\mathrm{Re}(\tilde{A}\tilde{B}^*)$

where $\tilde{A} \equiv A_0 e^{i\phi_A}$ $\quad A(t) = \mathrm{Re}(\tilde{A} e^{-i\omega t})$
$\tilde{B} = B_0 e^{i\phi_B}$ $\quad B(t) = \mathrm{Re}(\tilde{B} e^{-i\omega t})$

Proof: There are lots of way to do this, here's one of them

$$A(t) = \mathrm{Re}(\tilde{A} e^{-i\omega t}) = \tilde{A}' \cos\omega t + \tilde{A}'' \sin\omega t$$
$$B(t) = \mathrm{Re}(\tilde{B} e^{-i\omega t}) = \tilde{B}' \cos\omega t + \tilde{B}'' \sin\omega t$$

(where $\tilde{A} = \tilde{A}' + i\tilde{A}''$, $\tilde{B} = \tilde{B}' + i\tilde{B}''$)

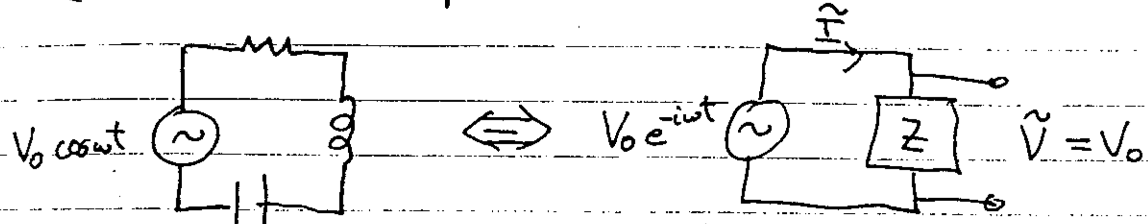
$$\therefore \langle A(t) B(t)\rangle = \tilde{A}'\tilde{B}' \langle \cos^2\omega t\rangle + \tilde{A}''\tilde{B}''\langle \sin^2\omega t\rangle + (\tilde{A}'\tilde{B}'' + \tilde{A}''\tilde{B}')\langle \cos\omega t \sin\omega t\rangle$$

Now $\langle \cos^2\omega t\rangle = \langle \sin^2\omega t\rangle = 1/2$
$\langle \cos\omega t \sin\omega t\rangle = \frac{1}{2}\langle \sin 2\omega t\rangle = 0$
} check these

$$\Rightarrow \langle A(t) B(t)\rangle = \frac{1}{2}(\tilde{A}'\tilde{B}' + \tilde{A}''\tilde{B}'')$$
$$= \frac{1}{2}\mathrm{Re}(\tilde{A}\tilde{B}^*) = \frac{1}{2}\mathrm{Re}(\tilde{A}^*\tilde{B})$$

QED

(b) Power dissipated in an RLC circuit



$$Z = R - i\left(\omega L - \frac{1}{\omega C}\right)$$

From part (a)

$$\langle P(t) \rangle = \langle V(t)\, I(t) \rangle = \frac{1}{2} \mathrm{Re}\left(\tilde{V}^* \tilde{I}\right)$$

Here $\tilde{V}$ and $\tilde{I}$ are the complex amplitudes of the voltage and current across the load

$$\tilde{V} = V_0 \qquad \tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{Z}$$

$$\therefore \langle P \rangle = \frac{1}{2} \mathrm{Re}\left(\frac{V_0^2}{Z}\right) = \frac{V_0^2}{2} \mathrm{Re}\left(\frac{1}{R - i(\omega L - \frac{1}{\omega C})}\right)$$

$$= \frac{V_0^2}{2} \mathrm{Re}\left(\frac{R + i(\omega L - \frac{1}{\omega C})}{R^2 + (\omega L - \frac{1}{\omega C})^2}\right)$$ Multiplying top and bottom by $R + i(\omega L - \frac{1}{\omega C})$

$$= \frac{V_0^2}{2} \frac{R}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$= \frac{V_0^2}{2R} \frac{\frac{R^2}{L^2}\omega^2}{\omega^2 \frac{R^2}{L^2} + (\omega^2 - \frac{1}{LC})^2}$$ (multiplying top and bottom by $\frac{\omega^2}{L^2}$)

$$= \frac{V_0^2}{2R} \frac{\Gamma^2 \omega^2}{\omega^2 \Gamma^2 + (\omega^2 - \omega_0^2)}$$

where $\Gamma \equiv \frac{R}{L}$ $\quad \omega_0 = \frac{1}{LC}$

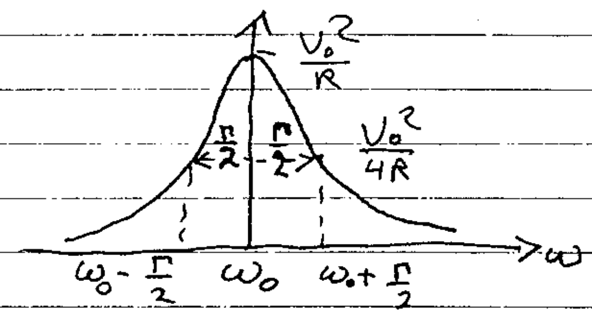
Now in the near resonance approximation

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega(\omega - \omega_0)$$

$$\Rightarrow \langle P \rangle = \frac{V_0^2}{2R} \frac{\Gamma^2 \omega^2}{\omega^2 \Gamma^2 + 4\omega^2(\omega - \omega_0)^2}$$

$$\boxed{\langle P \rangle = \frac{V_0^2}{2R} \frac{\Gamma^2/4}{(\omega - \omega_0)^2 + \frac{\Gamma^2}{4}}}$$

Near the resonance, the ⟨Power⟩ dissapated looks lik
a Lorentzian



$\text{FWHM} = \Gamma$

(c) The average stored energy is the sum of energies in C +

$$\langle E \rangle = \langle E_c \rangle + \langle E_L \rangle = \frac{1}{2} C \langle V_c^2 \rangle + \frac{1}{2} L \langle I_L^2 \rangle$$

But $\langle E_c \rangle = \langle E_L \rangle$ (capacitor + inductor store equal energies)

$$\Rightarrow \langle E \rangle = 2\langle E_L \rangle = L \langle I_L^2 \rangle$$

$$= L\left(\frac{Re}{2}\langle \tilde{I}\tilde{I}^* \rangle\right) = \frac{L}{2} Re\left(\frac{\tilde{V}}{Z}\frac{\tilde{V}^*}{Z^*}\right)$$

$$= \frac{L}{2}\frac{V_0^2}{|Z|^2}$$

On resonance $Z = R \Rightarrow \langle E(\omega = \omega_0) \rangle = \frac{L V_0^2}{2R^2}$

On resonance, the average power dissipated is

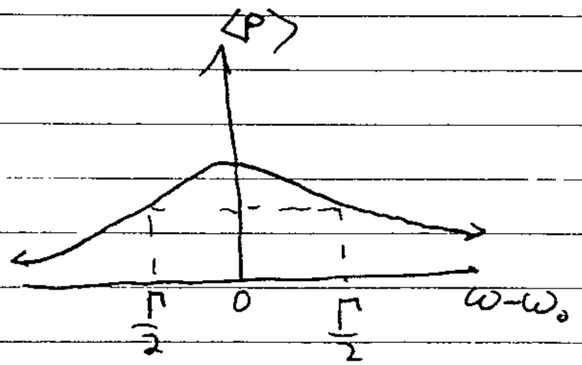
$$\langle P \rangle \big|_{\omega=\omega_0} = \frac{V_0^2}{2R}$$

$$\therefore \; Q \equiv \omega_0 \frac{\langle E \rangle}{\langle P \rangle} = \omega_0 \frac{\frac{L V_0^2}{2R^2}}{\frac{V_0^2}{2R}}$$

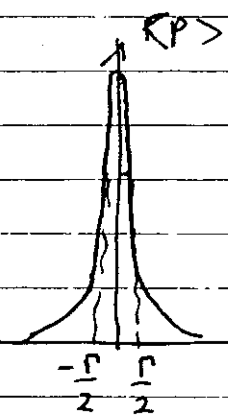
$$\Longrightarrow \boxed{Q = \omega_0 \frac{L}{R} = \frac{\omega_0}{\Gamma}}$$

This is the general form of Q: ratio of oscillation frequency to the damping rate.

It measures the number of periods oscillated before the stored energy decays to 1/e of its value. For a good oscillator Q is very large ($\sim 10^4$). The resonance is then very narrow (i.e. if we want to deliver power to the load we must be almost exactly on resona ;

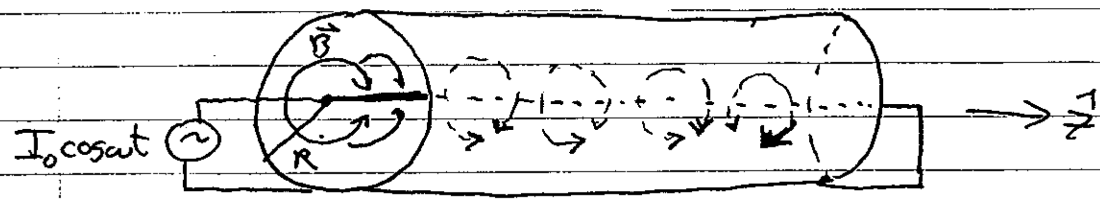


Low Q
(R big)



High Q
(R small)

Problem 3   Griffiths 7.16



7.16 (a) The magnetic field lines point in the azimuthal direction (alternating from $+\hat{\phi}$ to $-\hat{\phi}$)

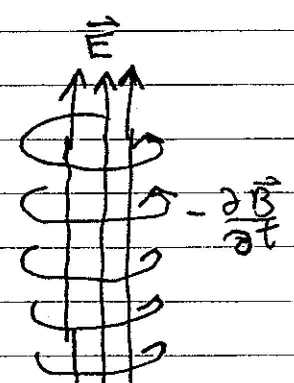
Remember that Faraday's Law for $\vec{E}$ is like Ampere's Law for $\vec{B}$ with magnetic flux playing the role of current density:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Updownarrow \qquad \Updownarrow$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

The magnetic flux lines are changing in the $\hat{\phi}$ direction. This is like current going in the $\hat{\phi}$ direction i.e. a SOLENOID! Thus, $\vec{E}$ points in the $\hat{z}$ direction

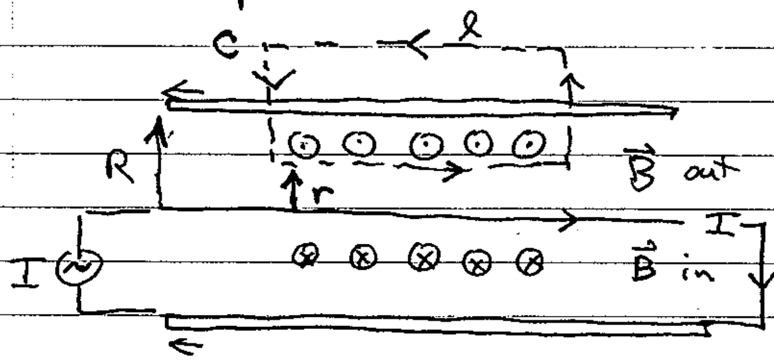


Also from symmetry the magnitude of $\vec{E}$ depends only on the distance from the axis r

$$\Rightarrow \boxed{\vec{E} = E(r)\,\hat{z}}$$

7.16 (b) Look at the coaxial cable in cross-section.

- For a effectly infinite cylinder, $\vec{E}$ is zero outside $r > R$ as for a solenoid.
- The $\vec{B}$ field is zero for $r > R$ from ampere's Law



At some instant of time the field will be in such a configurati[on]

Now take Faraday's Law around contour C

$$\oint_C \vec{E}\cdot d\vec{\ell} = \underbrace{E(r)\,\ell}_{\text{bottom leg}} + \underbrace{0 + 0}_{\substack{\text{side legs}=0\\ \text{because } \vec{E}\perp d\vec{\ell}}} + 0$$

$\leftarrow$ top leg $= 0$ because $|\vec{E}| = 0$ for $r > R$

$$\Phi_B = \int_S \vec{B}\cdot d\vec{a} = \ell \int_r^R dr'\, B(r') = \ell \int_r^R dr' \frac{\mu_0 I(t)}{2\pi r'}$$

$$= \ell \frac{\mu_0 I(t)}{2\pi} \ln\left(\frac{R}{r}\right) = \ell \frac{\mu_0 I_0 \cos\omega t}{2\pi} \ln\left(\frac{R}{r}\right)$$

$$\therefore\quad -\frac{d\Phi_B}{dt} = +\ell \frac{\mu_0 \omega I_0}{2\pi} \sin\omega t \ln\left(\frac{R}{r}\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{\mu_0 \omega I_0}{2\pi} \sin\omega t \ln\left(\frac{R}{r}\right) \hat{z}}$$