Physics 406: P.S#4 Solutions

(Griffiths 7.16)

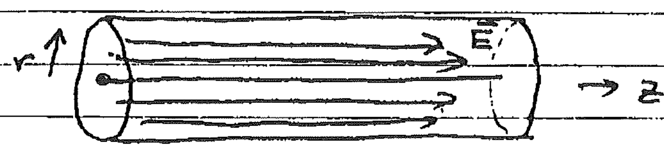
The previous problem gave the solution in the "quasi-static" regime. That is to say, we calculated $\vec{B}$ from Ampère's Law excluding any displacement-current, and then used this to find the induced $\vec{E}$ via Faraday's Law. This will be valid as long as

$$\boxed{L \ll cT}$$

where L is any characteristic dimension of interest and T is the characteristic time in which currents are changing, c is the speed of light.

Now let's find the missing displacement-current.

In the geometry of the problem



$$\vec{E}(r,t) = \frac{\mu_0 I}{2\pi} \omega \sin \omega t \ln\left(\frac{R}{r}\right) \hat{z}$$

7.33 (a) The displacement current density

$$\vec{J}_d = \cancel{\mu_0}\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \boxed{\frac{\cancel{\mu_0} I}{2\pi} \frac{\omega^2}{c^2} \cos \omega t \ln\left(\frac{R}{r}\right) \hat{z}}$$

(Having used $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$)

7.33(b) The total displacement current is the flux of $\vec{J}_d$ through the area enclosed by the cylinder

$$I_d = \int \vec{J}_d \cdot d\vec{a} = \int_0^R \vec{J}_d \cdot (2\pi r\, dr\, \hat{z})$$

$$= I \frac{\omega^2}{c^2} \cos\omega t \int_0^R r \ln\left(\frac{r}{R}\right) dr$$

Aside: $r \ln\left(\frac{R}{r}\right) = \frac{1}{2}\frac{d(r^2)}{dr} \ln\left(\frac{R}{r}\right) = \frac{1}{2}\left(\frac{d}{dr}\left(r^2 \ln\left(\frac{R}{r}\right)\right) - r^2 \frac{d}{dr}\ln\left(\frac{R}{r}\right)\right)$

(Integration by parts) $= \frac{1}{2}\left(\frac{d}{dr}\left(r^2 \ln\left(\frac{R}{r}\right)\right) + r\right)$

$$\therefore \int_0^R r \ln\left(\frac{r}{R}\right) dr = \left[\frac{r^2}{2} \ln\left(\frac{R}{r}\right) + \frac{r^2}{4}\right]_0^R = \frac{R^2}{4}$$ (since ln diverges slower than r^2)

$$\therefore \boxed{I_d = \frac{1}{4} \frac{\omega^2 R^2}{c^2} I \cos\omega t}$$

7.33 (c) $\frac{I_d(t)}{I\cos\omega t} = \frac{1}{4}\frac{\omega^2 R^2}{c^2}$, Given $R = 2\text{ mm} = 2\times10^{-3}\text{ m}$

$$\frac{I_d}{I\cos\omega t} = 0.1 = \frac{1}{4}\frac{\omega^2 R^2}{c^2}$$

$$\Rightarrow \omega^2 = 0.4 \frac{c^2}{R^2}$$

$$\boxed{\begin{aligned} \omega &= \sqrt{0.4}\,\frac{c}{R} = 9.5\times10^{10} \text{ rad/sec} \\ \nu &= \frac{\omega}{2\pi} = 15 \text{ GHz} \end{aligned}}$$

A pretty big (but not unreasonal frequency)

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Comments

We see here that $\frac{I_d}{I_{conduction}} \sim \frac{\omega^2}{c^2} R^2 \sim \left(\frac{R}{cT}\right)^2 << 1$

As expected, the displacement current is negligible if the time it takes light to propagate across the dimension of the system is small compare to the ~~rate~~ time over which the current changes

$$\frac{R}{c} << T \quad \text{(Quasi static)}$$

Stated another way, we know that

$\frac{\omega}{c} = k = \frac{2\pi}{\lambda}$, where λ is the wavelength of the wave created by a signal oscillating at ω

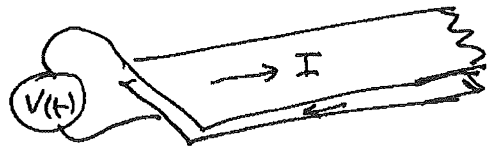
Thus $\frac{\omega^2}{c^2} R^2 \sim \frac{R^2}{\lambda^2} << 1$

$$R << \lambda \quad \text{(Quasi static)}$$

Of course this is all a big of a fudge since we assumed an <u>infinitely</u> long cylinder. This will only be self consistent if

$$L << \lambda \quad \text{i.e.} \quad \frac{L}{c} << T$$

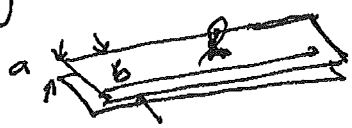
Problem 2: Parallel Plate Transmission Line



The problem here is to show how voltage and current propagate down a "transmission line"

Our first task is to determine the capacitance and inductance / length.

Let's give the line dimensions



The capacitance of parallel plate

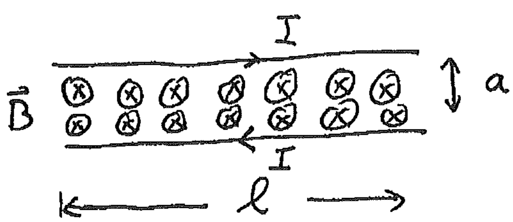
$$C = \epsilon_0 \frac{A}{a} = \epsilon_0 \frac{b\ell}{a}$$

$\Rightarrow$ $\frac{\text{Capacitance}}{\text{length}}$ $\quad \mathcal{C} = \frac{C}{\ell} = \boxed{\epsilon_0 \frac{b}{a}}$

To find inductance, we put current on the plates, and determine the flux through a surface. then

$$\Phi = L I$$

Edge on:



$\Rightarrow \Phi = B a \ell$

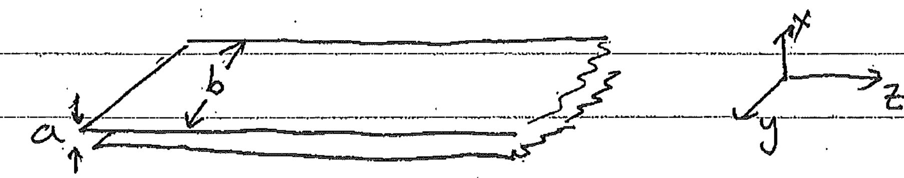
$B = \mu_0 K$ for parallel plate (see 405)

$K = \frac{I}{b}$

$$\Rightarrow \Phi = \mu_0 \frac{a\ell}{b} I \Rightarrow L = \mu_0 \frac{a\ell}{b}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{L}{\ell} = \mu_0 \frac{a}{b}}$$

Transmission Line



(a) Maxwell's equations: Between the plates $\rho = \vec{J} = 0$

We also know $\vec{E}(\vec{r},t) = E_x(z,t)\,\hat{x}$

$\vec{B}(\vec{r},t) = B_y(z,t)\,\hat{y}$

$$\Rightarrow \underbrace{\frac{\partial E_x}{\partial x} = 0}_{\text{Gauss}}, \quad \underbrace{\frac{\partial B_y}{\partial y} = 0}_{\text{no mag. monopoles}}, \quad \underbrace{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}}_{\text{Faraday}}, \quad \underbrace{\frac{\partial B_y}{\partial z} = -\mu_0\epsilon_0\frac{\partial E_x}{\partial t}}_{\text{Ampère}}$$

Now, the electric and magnetic fields are related to the local voltage and current

$$V = +\int_0^a E_x\,dx \Rightarrow E_x = \frac{V}{a}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B_y = \mu_0\left(\frac{I}{b}\right)$$ ← Surface current

Faraday's Law $\Rightarrow \frac{\partial}{\partial z}\left(\frac{V}{a}\right) = -\frac{\partial}{\partial t}\left(\frac{\mu_0 I}{b}\right)$

$$\Rightarrow \boxed{\frac{\partial V}{\partial z} = -\frac{\mu_0 a}{b}\frac{\partial I}{\partial t} = -\mathcal{L}\frac{\partial I}{\partial t}}$$

$$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\mu_0 I}{b}\right) = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{V}{a}\right)$$

$$\Rightarrow \boxed{\frac{\partial I}{\partial z} = -\frac{\epsilon_0 b}{a}\frac{\partial V}{\partial t} = -\mathcal{C}\frac{\partial V}{\partial t}}$$

(b) We want to derive the wave equation from the coupled set of first order p.D.E.s

- Take $\frac{\partial}{\partial z}$ of Faraday's Law

$$\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\mathcal{L}\frac{\partial I}{\partial t}\right)$$

$$\Downarrow$$

$$\frac{\partial^2 V}{\partial z^2} = -\mathcal{L}\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial z}\right)$$

[Having switched the order of derivatives]

$$\Downarrow$$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = +\mathcal{L}\mathcal{C}\,\frac{\partial^2 V}{\partial t^2}}$$

[Substituting for $\frac{\partial I}{\partial z}$ from Ampère's Law]

- Similarly, take $\frac{\partial}{\partial z}$ of Ampère's Law

$$\frac{\partial}{\partial z}\left(\frac{\partial I}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\mathcal{C}\frac{\partial V}{\partial t}\right)$$

$$\Downarrow$$

$$\frac{\partial^2 I}{\partial z^2} = -\mathcal{C}\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial z}\right)$$

$$\Downarrow$$

$$\boxed{\frac{\partial^2 I}{\partial z^2} = \mathcal{L}\mathcal{C}\,\frac{\partial^2 I}{\partial t^2}}$$

Thus the local voltage and current satisfy the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f = 0$$

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Comparing the wave eq. for V and I with the general form, we find the phase velocity

$$v = \frac{1}{\sqrt{\mathcal{L}C}} = \frac{1}{\sqrt{\left(\frac{\mu_0 a}{b}\right)\left(\frac{\epsilon_0 b}{a}\right)}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c!$$

Thus the voltage and current waves propagate at the speed of light!

(c) Let's plug in the trial solutions

$$V(z,t) = V_0 \cos(kz - \omega t) \qquad I(z,t) = I_0 \cos(kz - \omega t)$$

$$\frac{\partial^2 V}{\partial z^2} = -k^2 V \qquad \frac{\partial^2 V}{\partial t^2} = -\omega^2 V$$

$$\Rightarrow -k^2 + \frac{\omega^2}{\mathcal{L}C} = 0$$

Thus our ansatz is a solution iff

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mathcal{L}C}} = c \quad \text{the phase velocity}$$

The same is true for the current wave

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Now, the amplitude of the voltage and current waves are not independant. They are coupled by Maxwell's eqn's:

$$\frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t} \Rightarrow -kI_o \sin(kz-\omega t) = -\omega \mathcal{C} V_o \times \sin(kz-\omega t)$$

$$\Rightarrow V_o = \frac{k}{\omega}\frac{1}{\mathcal{C}} I_o$$

But $\frac{\omega}{k} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}$ $\Rightarrow$ $\boxed{V_o = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\, I_o}$

Thus the Voltage and current amplitudes satifies an "Ohm's Law" with the proportionality constant know as the "characteristic impedance" of the line

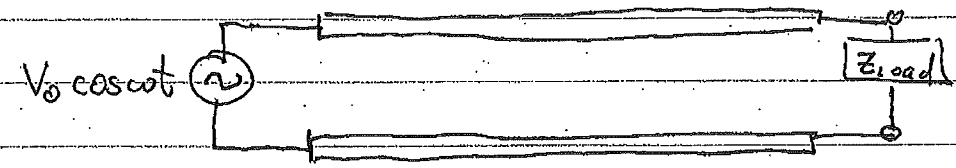
$$Z_c = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \sqrt{\frac{\mu_o \frac{a}{b}}{\epsilon_o \frac{b}{a}}} = \sqrt{\frac{\mu_o}{\epsilon_o}}\,\frac{a}{b}$$

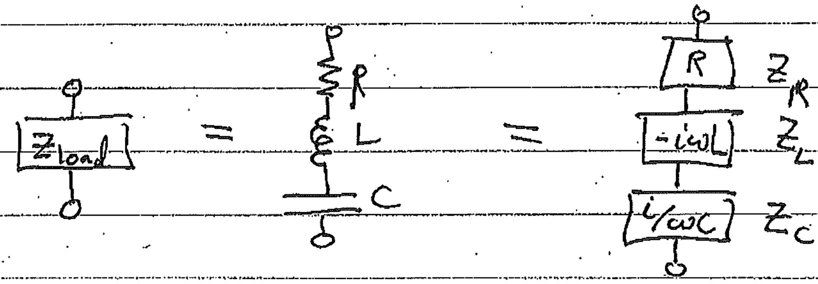
$$\Rightarrow \boxed{Z_c = Z_o \frac{a}{b}}$$

where $Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}\ N/A^2}{8.85\times 10^{-12}\ \frac{coul}{Volt\text{-}m}}} = 377$ ohms

This is known as the impedance of free space

(d) Now a load terminates the transmission line





$R = 3.77$ ohm

$L = 10^{-6}$ H

$C = 10^{-6}$ F

$$Z_{load} = R - i\omega L + \frac{i}{\omega C}$$

$$= R\left(1 - i\frac{\Gamma}{\omega}(\omega^2 - \omega_0^2)\right)$$

$\Gamma = \frac{L}{R}$ (damping) $\quad \omega_0 = \frac{1}{\sqrt{LC}}$ (resonance frequenc)

The characteristic impedance of the transmission line for the case $b = 100a$ ...

$$Z_c = \frac{a}{b} Z_0 = \frac{1}{100} Z_0 = 3.77 \text{ ohm}$$

⇒ The load impedance will match the characteristic impedance if

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}} = 10^6 \text{ s}^{-1}$$

resonance

Problem 3

Given $\vec{E}(\vec{r},t) = (\hat{x} + 2\hat{y} + E_z\hat{z})\cos(-3x + y + z - \omega t)$

The general form of a plane wave (linearly polarized)

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$$\vec{k}\cdot\vec{r} = -3x + y + z \Rightarrow \vec{k} = -3\hat{x} + \hat{y} + \hat{z}$$

$$\vec{E}_0 = \hat{x} + 2\hat{y} + E_z\hat{z}$$

(a) Direction of propagation $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$, $|\vec{k}| = \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}$

$$\therefore \boxed{\hat{k} = \frac{1}{\sqrt{11}}(-3\hat{x} + \hat{y} + \hat{z})}$$

(b) $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0 = (-3\hat{x} + \hat{y} + \hat{z})\cdot(\hat{x} + 2\hat{y} + E_z\hat{z})$

$$\therefore -3 + 2 + E_z = 0 \Rightarrow \boxed{E_z = 1 \ \frac{\text{Volts}}{m}}$$

Polarization is linear, along direction

$$\vec{\varepsilon} = \frac{\vec{E}_0}{|\vec{E}_0|} = \frac{1}{\sqrt{6}}(\hat{x} + 2\hat{y} - \hat{z})$$

(c) Wavelength: $\boxed{\lambda = \frac{2\pi}{|\vec{k}|} = \frac{2\pi}{\sqrt{11}} \ m}$

(d) Frequency $\nu = \frac{\omega}{2\pi} = \frac{ck}{2\pi} = \frac{c}{\lambda}$

$$\Rightarrow \nu = \frac{3\times10^8 \ m/s}{\frac{2\pi}{\sqrt{11}} \ m} = 1.6\times10^8 \ s^{-1} = 160 \ MHz$$

(TV broadcast)

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Problem 3 continued

(c) For a travelling plane wave, the magnetic field generally has the same form as the electric field.

$$\vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi) \quad \text{(in phase with electric field)}$$

with $\vec{k}\cdot\vec{B}_0 = 0$, $\vec{E}_0\cdot\vec{B}_0 = 0$, $\vec{E}_0 \times \vec{B}_0 \propto \vec{k}$

and $B_0 = E_0/c$

If the wave were propagating along one of the Cartesian axes, we could use these to simply write down the magnetic field. For this more complicated geometry it is easier to return directly to Maxwell's eqn.

Use complex representation: $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow i\vec{k}\times\vec{E}_0 = i\omega\vec{B}_0 \quad \left(\text{Remember } \vec{\nabla} \to +i\vec{k} \quad \frac{\partial}{\partial t} \to -i\omega\right)$$

$$\Rightarrow \vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{\vec{k}}{ck}\times\vec{E}_0$$

$$= \frac{1}{ck}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \frac{1}{ck}\left[-\hat{x} + 4\hat{y} - 7\hat{z}\right]$$

$$ck = \omega = 2\pi\nu = 3\sqrt{11}\times 10^8 \ s^{-1}$$

$$\Rightarrow \boxed{\vec{B}(\vec{r},t) = \frac{10^{-8}}{3\sqrt{11}}(-\hat{x} + 4\hat{y} - 7\hat{z})\cos(-3x + y + z - \omega t)}$$