Physics 406: Electricity and Magnetism
Problem Set #5 Solutions

(1) Standing E&M Wave

$\vec{k}_1 = k\hat{z}$ $\qquad\qquad$ $\vec{k}_2 = -k\hat{z}$

$$\vec{E}_1 = \hat{x}\, E_0 \cos(kz - \omega t) \qquad\qquad \vec{E}_2 = \hat{x}\, E_0 \cos(kz + \omega t)$$

(a) $$\vec{E}_3 = \vec{E}_1 + \vec{E}_2 = \hat{x}\, E_0 \big(\cos(kz - \omega t) + \cos(kz + \omega t)\big)$$

$$= \hat{x}\, E_0 \big(\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t) + \cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t) \big)$$

$$\Rightarrow \boxed{\vec{E}_3 = \hat{x}\, 2E_0 \cos(kz)\cos(\omega t)}$$

Alternative solution using complex representation

$$\tilde{\vec{E}}_1 = \hat{x}\, E_0 e^{i(kz - \omega t)}, \qquad \tilde{\vec{E}}_2 = \hat{x}\, E_0 e^{i(kz + \omega t)}$$

$$\tilde{\vec{E}}_3 = \tilde{\vec{E}}_1 + \tilde{\vec{E}}_2 = \hat{x}\, E_0 e^{ikz}\left(e^{-i\omega t} + e^{i\omega t}\right) = \hat{x}\, E_0 e^{ikz} (2\cos\omega t)$$

$$\vec{E}_3 = \mathrm{Re}(\tilde{\vec{E}}_3) = \hat{x}\, 2E_0 \left\{\mathrm{Re}(e^{ikz})\right\} \cos\omega t$$

$$= \hat{x}\, 2E_0 \cos(kz) \cos(\omega t)$$

(b) Since the total field is not a travelling wave we cannot use $\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E}$

However $\vec{B}_3$ is monochromatic

$$\Rightarrow \vec{B}_3 = \mathrm{Re}\left(\tilde{\vec{B}}_3(z) e^{-i\omega t}\right), \qquad \vec{E}_3 = \mathrm{Re}\left(\tilde{\vec{E}}_3(z)\, e^{-i\omega t}\right)$$

Plug this into $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \quad i\omega \underset{\sim}{\vec{B}}_3(z) = \vec{\nabla}\times \underset{\sim}{\vec{E}}_3(z) = \hat{y}\,\frac{\partial}{\partial z}\left(2E_0 \cos kz\right)$$

$$= 2kE_0 \sin kz\ \hat{y}$$

$$\therefore \quad \underset{\sim}{\vec{B}}_3(z) = +2i\frac{k}{\omega} E_0 \sin kz\ \hat{y} = +2i\,\frac{E_0}{c}\sin kz\ \hat{y}$$

$$\therefore \quad \vec{B}_3(z,t) = Re\left(\underset{\sim}{\vec{B}}_3(z)\, e^{-i\omega t}\right) = 2\frac{E_0}{c}\sin kz\ Re\left(i\, e^{-i\omega t}\right)\hat{y}$$

$$\Rightarrow \vec{B}_3(z,t) = 2\,\frac{E_0}{c}\sin kz \sin\omega t\ \hat{y}$$

Alternative solution

Find $\vec{B}_1$ and $\vec{B}_2$ (the mag-field of the two plane waves)

$\vec{B}_1 = \hat{y}\,\frac{E_0}{c}\cos(kz-\omega t)$ since $\hat{k}_1 = \hat{z}$

$\vec{B}_2 = -\hat{y}\,\frac{E_0}{c}\cos(kz+\omega t)$ since $\hat{k}_2 = -\hat{z}$

$$\vec{B}_3(z,t) = \vec{B}_1 + \vec{B}_2 = \hat{y}\,\frac{E_0}{c}\,Re\left(e^{i(kz-\omega t)} + e^{-i(kz+\omega t)}\right)$$

$$= \hat{y}\,\frac{E_0}{c}\,Re\left[\left(e^{ikz} - e^{-ikz}\right)e^{-i\omega t}\right] = \hat{y}\,\frac{E_0}{c}\,Re\left[2i\sin kz\, e^{-i\omega t}\right]$$

$$= \hat{y}\,\frac{E_0}{c}\,2\sin kz \sin\omega t$$

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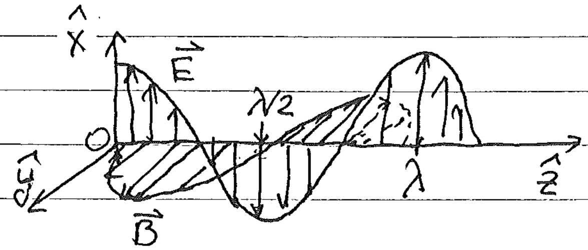
Thus, for a standing wave $\vec{E}$ and $\vec{B}$ are 90° out of phase, both in time and space

Problem 2 continued

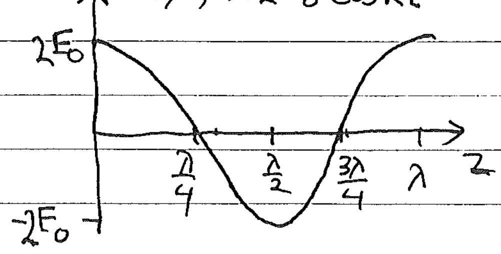


$\vec{E}$ and $\vec{B}$ at $t = T/8$

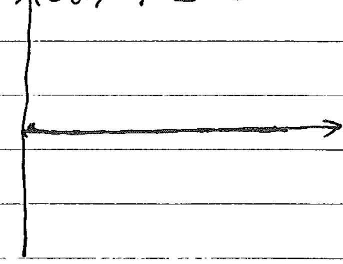
(c)

$t = 0$ and $t = T$

$E(z,0) = 2E_0 \cos kz$

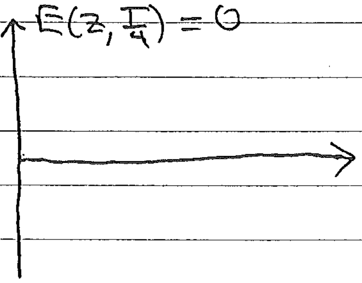


$B(z,0) = 0$

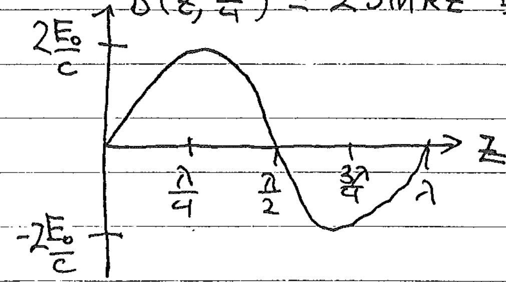


$t = \frac{T}{4}$

$E(z, \frac{T}{4}) = 0$

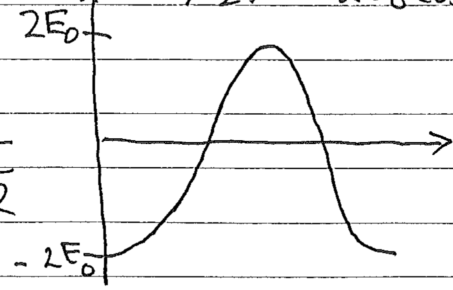


$B(z, \frac{T}{4}) = 2 \sin kz \; \frac{E_0}{c}$

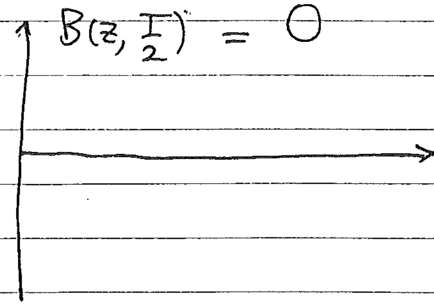


$t = \frac{T}{2}$

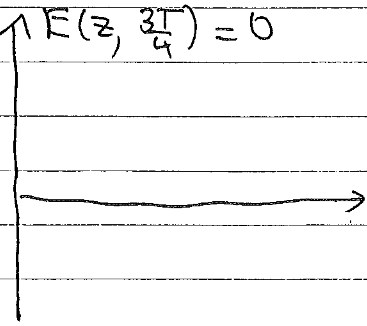
$E(z, \frac{T}{2}) = -2E_0 \cos kz$



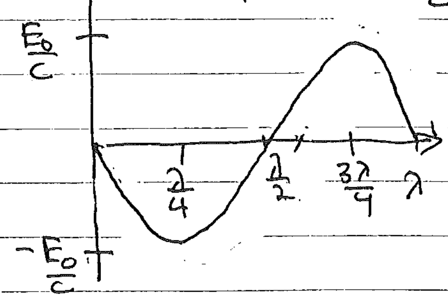
$B(z, \frac{T}{2}) = 0$



$E(z, \frac{3T}{4}) = 0$



$B(z, \frac{3T}{4}) = -2\frac{E_0}{c} \sin kz$



Anti-nodes of E at $\frac{m\lambda}{2}$

nodes of E at $\frac{\lambda}{2}(m+\frac{1}{2})$

Anti-nodes of B at $(m+\frac{1}{2})\frac{\lambda}{2}$

nodes of B at $m\frac{\lambda}{2}$

$m = 0, \pm 1$

Problem 1 continued

(d) Energy densities:

$$U_E = \frac{\epsilon_0}{2}|\vec{E}(z,t)|^2 = \frac{\epsilon_0}{2} 4E_0^2 \cos^2(kz)\cos^2(\omega t)$$

$$U_B = \frac{\epsilon_0}{2}|\vec{B}(z,t)|^2 = \frac{1}{2\mu_0}\, 4\frac{E_0^2}{c^2} \sin^2(kz)\sin^2(\omega t)$$

$$= \frac{\epsilon_0}{2}\, 4E_0^2 \sin^2(kz)\sin^2(\omega t)$$

$$\langle U_E\rangle = 2\epsilon_0 E_0^2 \cos^2(kz)\langle \cos^2(\omega t)\rangle = \epsilon_0 E_0^2 \cos^2 kz$$

$$\langle U_B\rangle = 2\epsilon_0 E_0^2 \sin^2 kz \langle \sin^2(\omega t)\rangle = \epsilon_0 E_0^2 \sin^2 kz$$

$$\langle U_{total}\rangle = \langle U_E\rangle + \langle U_B\rangle = \epsilon_0 E_0^2(\cos^2 kz + \sin^2 kz) = \epsilon_0 E_0$$

(e) $I = \langle|\vec{S}|\rangle = \frac{1}{\mu_0}\langle|\vec{E}\times\vec{B}|\rangle = \frac{1}{\mu_0}\frac{E_0^2}{c} \sin kz \cos kz \langle \sin\omega t \cos\omega t\rangle$

but $\langle \sin\omega t\cos\omega t\rangle = 0 \Rightarrow \boxed{I = 0}$

Interpretation: In a standing wave there is no flux of energy, as in a travelling wave. Instead the energy is oscillating back and forth between the electric and magnetic fields as in an LC circuit oscillator

(b) Let $\tilde{\Psi}(r) = \frac{u(r)}{r}$ (u is known as the "reduced radial" wave function)

$$\frac{\partial \tilde{\Psi}}{\partial r} = \frac{u'}{r} - \frac{u}{r^2} \quad \text{where} \quad u' = \frac{du}{dr}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\Psi}}{\partial r}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r u' - u\right]$$

$$= \frac{1}{r^2}\left[r^2 u'' + u' - u'\right] = \frac{u''}{r} = \frac{1}{r}\frac{d^2 u}{dr^2}$$

$$= -k^2 \tilde{\Psi} = -k^2 \frac{u(r)}{r}$$

$$\Rightarrow \boxed{\frac{d^2 u}{dr^2} + k^2 u = 0}$$

(c) This is nothing other than the simple harmonic oscillator equation.

General solution $\quad u(r) = u_0 \, e^{\pm ikr} e^{i\phi}$

u_0 and ϕ arbitrary

$$\Rightarrow \Psi(r,t) = \mathrm{Re}\left(\frac{u_0}{r} e^{\pm i(kr \pm \omega t + \phi)}\right)$$

$$\Rightarrow \boxed{\Psi = u_0 \frac{\cos(kr \pm \omega t + \phi)}{r}}$$

These are spherical waves (Ψ constant on a sphere)

$\Psi_1 \Rightarrow$ outward propagation, $\Psi_2 \Rightarrow$ inward propagation

Problem 2

Spherical Waves

Wave equation for a scalar field $\psi(\vec{r},t)$

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

(a) Monochromatic field whose amplitude depends only on the radial distance $r = |\vec{r}|$

$$\psi(\vec{r},t) = \tilde{\psi}(r)\, e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \tilde{\psi}(r)\, e^{-i\omega t}$$

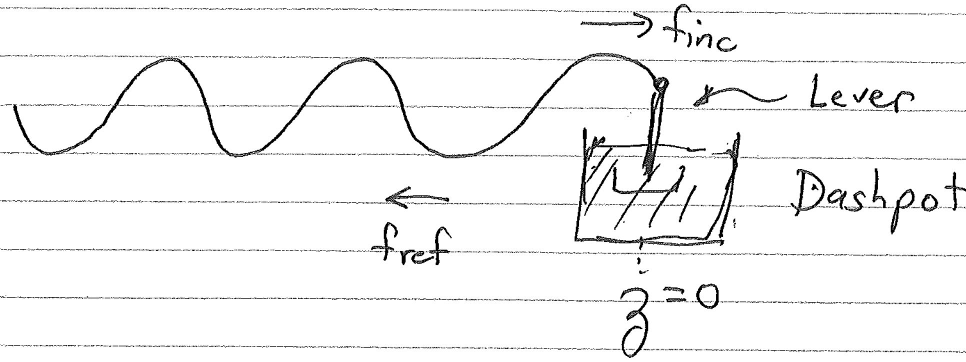
$$\nabla^2 \psi = e^{-i\omega t}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + \cancel{\partial_\theta^2(\cdots)}^{\,0} + \cancel{\partial_\phi^2(\cdots)}^{\,0}\right)$$

No θ or ϕ dependance

$$\Rightarrow \frac{e^{-i\omega t}}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + \frac{\omega^2}{v^2}\tilde{\psi}(r)\, e^{-i\omega t} = 0$$

$$\rightarrow \boxed{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + k^2 \tilde{\psi}(r) = 0 \quad \text{where } k = \frac{\omega}{v}}$$

Problem 3 : Impedance Mismatch



The wave traveling on the string is superposition of the incident wave traveling in the $+z$ direction $f_{inc}(z,t)$, and the reflected wave traveling in the $-z$ direct, $f_{ref}(z,t)$

The transverse force exerted by the string on the lever is interpreted as due to wave impedance

$$F_{\substack{\text{on lever}\\ \text{due to string}}} = -F_{\substack{\text{on string}\\ \text{due to lever}}} = -\left(-Z_1 \left.\frac{\partial f_{inc}}{\partial t}\right|_{z=0} + Z_1 \left.\frac{\partial f_{ref}}{\partial t}\right|_{z=0}\right)$$

$$= Z_1\left(\left.\frac{\partial f_{inc}}{\partial t}\right|_{z=0} - \left.\frac{\partial f_{ref}}{\partial t}\right|_{z=0}\right)$$

Since the dashpot has negligible mass, by Newton's law

$$F_{\substack{\text{on lever}\\ \text{due to string}}} = -F_{\substack{\text{on lever}\\ \text{due to dashpot}}} = -\left(-Z_2 \frac{\partial f_{total}}{\partial t}\right)$$

where $f_{total} = \left.(f_{inc} + f_{ref})\right|_{z=0}$

$$\Rightarrow Z_1 \left(\left.\frac{\partial f_{inc}}{\partial t}\right|_{z=0} - \left.\frac{\partial f_{ref}}{\partial t}\right|_{z=0} \right) = Z_2 \left(\left.\frac{\partial f_{inc}}{\partial t}\right|_{z=0} + \left.\frac{\partial f_{ref}}{\partial t}\right|_{z=0} \right)$$

$$\Rightarrow (Z_1 + Z_2) \left.\frac{\partial f_{ref}}{\partial t}\right|_{z=0} = (Z_1 - Z_2) \left.\frac{\partial f_{inc}}{\partial t}\right|_{z=0}$$

- Integrating with time

$$\Rightarrow (Z_1 + Z_2)\, f_{ref}\big|_{z=0} = (Z_1 - Z_2)\, f_{inc}\big|_{z=0}$$

$$\Rightarrow \boxed{\frac{f_{ref}\big|_{z=0}}{f_{inc}\big|_{z=0}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}}$$

Reflection is due to an impedance mismatch