Physics 406
Solutions: P.S. #6

Problem 1 Wave propagation in a dielectric

$$\vec{E} = \vec{E}_0 \cos(\sqrt{6}\, z - 6\times10^{10}\, t)$$ Wave propagating in z-direction

z in cm, t in sec

(a) The frequency of this wave is $\omega = 6\times10^{10}\, s^{-1}$

or $\nu = \frac{\omega}{2\pi} = \boxed{9.5\times10^{9}\ Hz}$.

This is in the microwave part of the spectrum

(b) The wave number $k = \sqrt{6}\ cm^{-1}$

the wave velocity is $v = \frac{\omega}{k} = \sqrt{6}\times10^{10}\, cm/s$

$$v = \frac{c}{n} \Rightarrow n = \frac{c}{v} = \frac{3\times10^{10}\ cm/s}{\sqrt{6}\times10^{10}\ cm/s} = \boxed{1.22}$$

(c) The vacuum wavelength is determined by the frequency: $\lambda_{vacuum} = \frac{c}{\nu} = 2\pi\frac{c}{\omega} = \boxed{3.1\ cm}$

Inside the medium $\lambda = \frac{\lambda_{vac}}{n} = \frac{2\pi}{k} = 2.56\ cm$

(d) The magnetic field for a plane wave $\vec{B} = \frac{\vec{k}}{\omega}\times\vec{E}$

$$\Rightarrow \vec{B} = \hat{k}\times\frac{\vec{E}}{v} = \hat{z}\times\frac{\vec{E}}{c/n}$$

$$\therefore \boxed{\vec{B} = n\,\hat{z}\times\frac{\vec{E}_0}{c}\cos(\sqrt{6}\, z - 6\times10^{10}\, t)}$$

Problem 2 Poynting's Theorem

Start with Maxwell's Eqns "in media"

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

For "linear" media $\vec{D} = \epsilon\vec{E}$ $\quad \vec{B} = \vec{H}/\mu$

where $\epsilon/\epsilon_0 = 1+\chi_e$, $\mu/\mu_0 = 1+\chi_m$

Let $\vec{S} = \vec{E}\times\vec{H}$ (the poynting vector)

$$\vec{\nabla}\cdot\vec{S} = \vec{\nabla}\cdot(\vec{E}\times\vec{H}) \underset{\Uparrow}{=} \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

vector identity "product rule"

Now substitute for $\vec{\nabla}\times\vec{E}$ from Faraday's Law
+ $\vec{\nabla}\times\vec{H}$ from Ampère's Law

$$\Rightarrow \vec{\nabla}\cdot\vec{S} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{J}_{free} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{S} + \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) = -\vec{J}_{free}\cdot\vec{E}$$

Linear case: $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \epsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\epsilon}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E})$

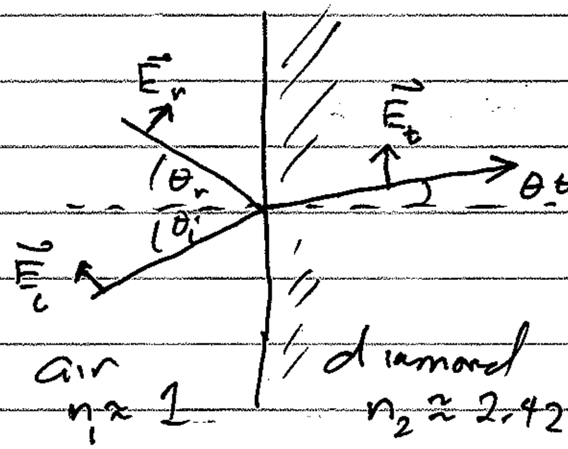
$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2\mu}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H})$

$$\Rightarrow \boxed{\begin{gathered}\vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} = -\vec{J}_{free}\cdot\vec{E} \\ u = \frac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})\end{gathered}}$$

Problem 3: Griffiths 9.17

Diamond's index of refraction $n = 2.42$

Reflection for polarization in the plane of incidence



$\mu_1 \cong \mu_2 \cong \mu_0$

$$r = \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

$n_1 \sin\theta_i = n_2 \sin\theta_t$
(Snell's Law)

(b) Brewster's angle: $\tan\theta_B = \frac{n_2}{n_1} \Rightarrow \theta_B = \tan^{-1}(2.42)$

$$\Rightarrow \boxed{\theta_B \cong 67.5^\circ}$$

(a) At normal incidence $r = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 2.42}{1 + 2.42}$

$$\Rightarrow \boxed{r = -0.415} \qquad \boxed{t = 1 + r = 0.585}$$

(c) Cross-over angle $(r=t)$

Using the boundary condition for $E_{\parallel}$

$$E_{i0}\cos\theta_i + E_{r0}\cos\theta_r = E_{t0}\cos\theta_t$$

$$\Rightarrow \cos\theta_i\,(1+r) = \cos\theta_t\, t \quad \left(\text{Having used } \theta_i = \theta_r\right)$$

@ cross-over $\theta_i \equiv \theta_c \qquad r = t$

Following Griffiths, define $\alpha \equiv \dfrac{\cos\theta_t}{\cos\theta_i}$

$$\frac{\cos\theta_t}{\cos\theta_c} = \frac{1+r}{r} = 1+\frac{1}{r} = \alpha_c$$

Now $r = \dfrac{\alpha-\beta}{\alpha+\beta}$ where $\beta \equiv \dfrac{n_2}{n_1}$

$$\Rightarrow 1+\frac{\alpha_c+\beta}{\alpha_c-\beta} = \frac{2\alpha_c}{\alpha_c-\beta} = \alpha_c \Rightarrow \boxed{\alpha_c = 2+\beta}$$

$$\alpha_c = \frac{\sqrt{1-\sin^2\theta_c/\beta^2}}{\cos\theta_c} = \frac{\sqrt{1-\sin^2\theta_c/\beta^2}}{\sqrt{1-\sin^2\theta_c}} = 2+\beta$$

$$\Rightarrow 1-\frac{\sin^2\theta_c}{\beta^2} = (2+\beta)^2\left(1-\sin^2\theta_c\right)$$

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Solve for $\sin^2\theta_c$

$$\Rightarrow \sin^2\theta_c\left(\frac{1}{\beta^2}-(2+\beta)^2\right) = 1-(2+\beta)^2$$

$$\Rightarrow \sin\theta_c = \sqrt{\frac{(2+\beta)^2-1}{(2+\beta)^2-\frac{1}{\beta^2}}}$$

Here $\beta \equiv \frac{n_2}{n_1} \cong 2.42$

$$\Rightarrow \sin\theta_c = \sqrt{\frac{18.53}{19.36}} = 0.978$$

$$\Rightarrow \boxed{\theta_c \cong 78.06^\circ}$$