Physics 406 Electricity and Magnetism II

Problem Set #7: DUE Friday Oct. 26 2012

Problem 1: Energy in Absorbing Dielectrics (10 points)

Consider a plane wave propagating in the z-direction in a (nonmagnetic) linear dielectric described by a complex index of refraction $\tilde{n}(\omega)$ and k-vector $\tilde{\mathbf{k}} = \tilde{k}\,\hat{\mathbf{z}} = \tilde{n}(\omega)\dfrac{\omega}{c}\,\hat{\mathbf{z}}$.

We showed in class that the real electric field is $\mathbf{E}(z,t) = \hat{\mathbf{x}}\,E_0 e^{-k_I z}\cos(k_R z - \omega t)$, where k_R, k_I are the real and imaginary parts of $\tilde{k}$, respectively.

(a) The real instantaneous energy Poynting vector is $\mathbf{S}(z,t) = \mathbf{E}(z,t) \times \mathbf{B}(z,t)/\mu_0$.

Show that the intensity is $\langle \mathbf{S}(z) \rangle = \dfrac{c\varepsilon_0}{2} n_R(\omega) E_o^2\, e^{-2k_I z}\,\hat{\mathbf{z}}$.,

where $n_R(\omega)$ is the real part of the index of refraction

(b) We have shown that the imaginary part of the index of refraction gives rises to attenuation of the wave propagating in a dielectric. The wave is attenuated because work is being done on the bound charges which is not returned to the wave. Show that the rate at which work is done on a bound charge (in steady state) by a monochromatic plane wave is

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2}\mathrm{Im}(\tilde{\alpha})\,\omega E_0^2,$$

where $\tilde{\alpha}$ is the complex polarizability, and E_0 is field amplitude.

Problem 2: Waves in Plasmas (10 Points)

Consider a neutral plasma (gas of electrons and positive ions), with electron density N_e. At high frequencies, because the ions are very heavy, we can consider them to be essentially fixed and any current due solely to the light electrons. The total charge density can be set to zero for an electrically neutral gas.

(a) Use Maxwell's Equations to derive the wave equation for the electric field,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \mu_0 \frac{\partial \mathbf{J}}{\partial t}.$$

(b) For a monochromatic wave, $\tilde{\mathbf{E}} = \mathrm{Re}(\tilde{\mathbf{E}} e^{-i\omega t})$, and ignoring any collision between electrons show that

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{E}} = \frac{\omega_p^2}{c^2}\tilde{\mathbf{E}}, \text{ where } \omega_p = \sqrt{N_e e^2/\varepsilon_0 m_e} \text{ is the “plasma frequency”}$$

(c) Derive the dispersion relation $k = \sqrt{\omega^2 - \omega_p^2}/c$. Sketch the graph $\omega(k)$.

(d) Give the real electric field when $\omega<\omega_p$.

(e) What is the reflection coefficient R of a wave traveling from vacuum into a plasma at normal incidence? What is the value of R when $\omega<\omega_p$.

Problem 3. Drude model of a conductor. (10 Points)

(a) We have taken the conductivity to be a constant, but in fact it is a function of the frequency of the excitation. One way to model this simply is due to Drude. We treat the conduction electrons and as free particles that undergo friction due to collisions at a rate γ and are driven by an oscillating field at frequency ω. Show that in that case, the conductivity, defined by $\tilde{\mathbf{J}}(\omega) = \tilde{\sigma}(\omega)\mathbf{E}$ is given by

$$\tilde{\sigma}(\omega) = \frac{\sigma_{DC}}{1 - i\omega/\gamma},$$

where $\sigma_{DC} = \varepsilon_0 \omega_p^2/\gamma$ is the DC conductivity and ω_p is the plasma frequency.

(b) Consider silver with $\sigma_{DC} = 6.17\times 10^{17}/(\mathrm{ohm\,m})$ and $\gamma = 2.5\times 10^{13}\,s^{-1}$. Give an approximate expression for the real and imaginary parts of the index of refraction at a typical microwave, visible, and x-ray frequency (take ε=1, μ=1). What is the skin depth of silver at a radio frequency?