Physic 406

Problem Set #7 Solutions

(1) Energy in Absorbing Dielectrics (10 points)

Given $\vec{E}(z,t) = \hat{x}\, E_0\, e^{-k_I z} \cos(k_R z - \omega t) = \hat{x}\, \mathrm{Re}\left(\underset{\sim}{\vec{E}}(z) e^{-i\omega t}\right)$

where $\underset{\sim}{\vec{E}}(z) = E_0\, e^{i \underset{\sim}{k} z}\, \hat{x}$

$\underset{\sim}{k} = \underset{\sim}{n}(\omega) \frac{\omega}{c}$ $\qquad$ $\underset{\sim}{n}(\omega) = n_R(\omega) + i\, n_I(\omega)$

(a) We seek the Poynting vector. First find the magnetic field from Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \implies i\underset{\sim}{\vec{k}} \times \underset{\sim}{\vec{E}} = -(-i\omega)\underset{\sim}{\vec{B}}$$ where $\underset{\sim}{\vec{k}} = \underset{\sim}{k}\hat{z}$

where $\vec{B}(z,t) = \mathrm{Re}\left(\underset{\sim}{\vec{B}}(z)\, e^{-i\omega t}\right)$

$$\implies \underset{\sim}{\vec{B}} = \frac{\underset{\sim}{\vec{k}}}{\omega} \times \underset{\sim}{\vec{E}}$$

Now the intensity is $\langle \vec{S}(z,t) \rangle = \frac{1}{\mu_0} \langle \vec{E}(z,t) \times \vec{B}(z,t) \rangle$

$$\implies \langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left(\underset{\sim}{\vec{E}}^*(z) \times \underset{\sim}{\vec{B}}(z)\right) = \frac{1}{2\omega\mu_0} \mathrm{Re}\left(\underset{\sim}{\vec{E}}^* \times (\underset{\sim}{\vec{k}} \times \underset{\sim}{\vec{E}})\right)$$

Aside $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$\implies \langle \vec{S} \rangle = \frac{1}{2\omega\mu_0} \mathrm{Re}\left(\underset{\sim}{\vec{k}}\, |\underset{\sim}{\vec{E}}|^2 + \underset{\sim}{\vec{E}}\, \cancel{(\underset{\sim}{\vec{k}} \cdot \underset{\sim}{\vec{E}}^*)}^{\,0}\right)$$

$$= \frac{1}{2\omega\mu_0} |\underset{\sim}{\vec{E}}|^2\, \mathrm{Re}(\underset{\sim}{\vec{k}}) = \frac{\mathrm{Re}(\underset{\sim}{k})}{2\omega\mu_0} |\underset{\sim}{\vec{E}}|^2\, \hat{z} =$$

$$\langle \vec{S} \rangle = \frac{\mathrm{Re}(\underset{\sim}{n})\,\frac{\omega}{c}}{2\omega\mu_0}\,|\underset{\sim}{\vec{E}}|^2\,\hat{z} = \frac{n_R}{2c\mu_0}\,|\underset{\sim}{\vec{E}}|^2\,\hat{z}$$

Substitute $\frac{1}{\mu_0} = c^2\epsilon_0$ $\quad |\underset{\sim}{\vec{E}}|^2 = E_0^2 e^{-2k_I z}$

$$\Rightarrow \boxed{\langle \vec{S} \rangle = \frac{c\epsilon_0}{2}\,n_R(\omega)\,E_0^2\,e^{-2k_I z}\,\hat{z}}$$

(b) The [rate at which] work is done by the field on the charge

$$\frac{dW}{dt} = F v \quad \text{where} \quad F = eE(t) = \text{force on charge}$$
$$v(t) = \frac{dx}{dt} = \text{velocity of "}$$

Time average $\left\langle \frac{dW}{dt} \right\rangle = \langle eE(t)\,v(t) \rangle = \frac{1}{2}\mathrm{Re}(e\,\underset{\sim}{E}^*\,\underset{\sim}{v})$

Now $\underset{\sim}{v} = -i\omega\underset{\sim}{x} \Rightarrow \left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2}\mathrm{Re}(-i\omega e\underset{\sim}{x}\,\underset{\sim}{E}^*)$

$e\underset{\sim}{x} = \underset{\sim}{p}$ (induced dipole moment)

$\Rightarrow \underset{\sim}{p} = \underset{\sim}{\alpha}\,\underset{\sim}{E}$ where $\underset{\sim}{\alpha}$ is the complex

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2}\mathrm{Re}(-i\omega\underset{\sim}{p}\,\underset{\sim}{E}^*) = \frac{1}{2}\mathrm{Re}(-i\omega\underset{\sim}{\alpha}\,|\underset{\sim}{E}|^2)$$
$$= \frac{1}{2}\omega E_0^2\,\mathrm{Re}(-i\underset{\sim}{\alpha}) = \boxed{\frac{1}{2}\mathrm{Im}(\underset{\sim}{\alpha})\,\omega E_0^2}$$

So the imaginary part of $\underset{\sim}{\alpha}$ is responsible for absorption

Problem 3 Waves in Plasmas

(a) In a neutral plasma $\rho = 0$
$$\vec{J} = eN_e \vec{v}_e$$

where N_e is the density of electrons
$\vec{v}_e$ is the velocity of the electrons
We assume the positive ions are fixed and do not contribute to the current. Also there are no other collisions with impurities

Maxwell's Eqns.

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$
$$= -\mu_0\frac{\partial \vec{J}}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = \mu_0\frac{\partial \vec{J}}{\partial t}}$$

Having used $\mu_0\epsilon_0 = \frac{1}{c^2}$

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(b) To get the dispersion relation we need the constitutive relation between $\vec{J}$ and $\vec{E}$.

Here are two ways to approach the problem

<u>Easy way:</u>

We have $\vec{J} = e N_e \vec{v}_e$. The equation of motion ~~xxxxx~~ for the electrons is simply Newton's law with the only force due to the external $\vec{E}$ field.

$$m_e \frac{d\vec{v}_e}{dt} = e\vec{E} \qquad \text{(no binding, or collisions)}$$

$$\Rightarrow \frac{d\vec{v}_e}{dt} = \frac{e}{m}\vec{E}$$

$$\Rightarrow \frac{\partial \vec{J}}{\partial t} = e N_e \frac{d\vec{v}}{dt} = \frac{e^2 N_e}{m}\vec{E} = \epsilon_0 \omega_p^2 \vec{E}$$

Plug into wave equation:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = \mu_0 \frac{\partial \vec{J}}{\partial t} = \mu_0 \epsilon_0 \omega_p^2 \vec{E} = \frac{\omega_p^2}{c^2}\vec{E}$$

For monochromatic waves $\vec{E} = \mathrm{Re}\left(\underset{\sim}{\vec{E}}(\vec{r})\, e^{-i\omega t}\right)$

$$\Rightarrow \left[\nabla^2 - \frac{1}{c^2}(-i\omega)^2\right]\underset{\sim}{\vec{E}} = \boxed{\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\underset{\sim}{\vec{E}} = \frac{\omega_p^2}{c^2}\underset{\sim}{\vec{E}}}$$

<u>Hard way</u>

Start with Ohm's Law $\underset{\sim}{\vec{J}} = \underset{\sim}{\sigma}\underset{\sim}{\vec{E}}$

$$\underset{\sim}{\sigma}(\omega) = \frac{\sigma_0}{1 - \frac{i\omega}{\gamma}} \qquad \sigma_0 = \epsilon_0 \frac{\omega_p^2}{\gamma} \quad \left(\begin{array}{l}\text{static}\\ \text{conductivity}\end{array}\right)$$

$\gamma =$ collision rate

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Now in a Plasma the collision rate $\gamma \to 0$

$$\Rightarrow \sigma(\omega) \approx \frac{\sigma_0}{-i\frac{\omega}{\gamma}} = i\,\epsilon_0 \frac{\omega_p^2}{\gamma}\frac{\gamma}{\omega} = i\,\epsilon_0 \frac{\omega_p^2}{\omega}$$

$$\Rightarrow \underset{\sim}{\vec{J}} = i\epsilon_0 \frac{\omega_p^2}{\omega} \underset{\sim}{\vec{E}}$$

In the "frequency domain"

$$\left[\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = \mu_0 \frac{\partial \vec{J}}{\partial t} \Rightarrow \left[\nabla^2 + \frac{\omega^2}{c^2}\right]\underset{\sim}{\vec{E}} = -i\omega\mu_0 \underset{\sim}{\vec{J}}$$

$$\Rightarrow \boxed{\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\underset{\sim}{\vec{E}} = \mu_0\epsilon_0\,\omega_p^2\,\underset{\sim}{\vec{E}} = \frac{\omega_p^2}{c^2}\underset{\sim}{\vec{E}}}$$

(c) To get the dispersion relation, plug in a plane wave solution

$$\underset{\sim}{\vec{E}}(\vec{r}) = \vec{E}_0\, e^{i\vec{k}\cdot\vec{r}} \qquad \vec{\nabla} \Rightarrow i\vec{k}$$

$$\Rightarrow -k^2 + \frac{\omega^2}{c^2} = \frac{\omega_p^2}{c^2}$$

$$\Rightarrow \boxed{k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}}$$

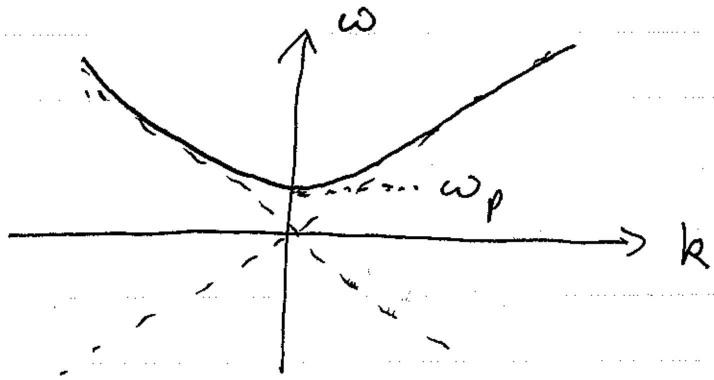
If $\omega < \omega_p$ k is pure imaginary

$\omega < \omega_p \Rightarrow k = i\beta$ where $\beta = \frac{\sqrt{\omega_p^2 - \omega^2}}{c}$ (real)

$$\Rightarrow \boxed{\vec{E} = \mathrm{Re}\left(\vec{E}_0\, e^{-\beta z}\, e^{-i\omega t}\right) = \vec{E}_0 e^{-\beta z} \cos\omega t}$$

This wave does not propagate into the plasma (evanescent wave

(c) Continued $\omega(k) = \sqrt{c^2k^2 + \omega_p^2}$



Dispersion relation

Note: $\frac{\omega}{k} = c\left(1 + \left(\frac{\omega_p}{k}\right)^2\right)^{1/2} > c$

(Phase velocity > c)

But $\frac{d\omega}{dk} = \frac{c}{\left(1 + \left(\frac{\omega_p}{k}\right)^2\right)^{1/2}} < c$

(Group velocity < c)

(d) When $\omega < \omega_p$

$$k(\omega) = \sqrt{\frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}} = i\underbrace{\sqrt{\frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2}}}_{\equiv \beta}$$

pure imaginary

$$\Rightarrow E(z,t) = \mathrm{Re}\left(E_0 e^{i(k(\omega)z - \omega t)}\right)$$

$$= \mathrm{Re}\left(E_0 e^{-\beta z} e^{-i\omega t}\right)$$

$$\boxed{E(z,t) = E_0 e^{-\beta z}\cos\omega t}$$

Purely decaying wave (No propagation)

"$\omega_p \equiv$ cutoff"

Evanescent Wave

(e) reflection coefficient

$$R = \left| \frac{Z_p - Z_0}{Z_p + Z_0} \right|^2 = \left| \frac{n_p - 1}{n_p + 1} \right|^2$$

where Z_p, n_p are the impedance, index of refraction of the plasma

and Z_0, n_0 are the impdence/ index of vacuum

$(n_0 = 1)$ (I assumed $\mu = \mu_0$ in plasma)

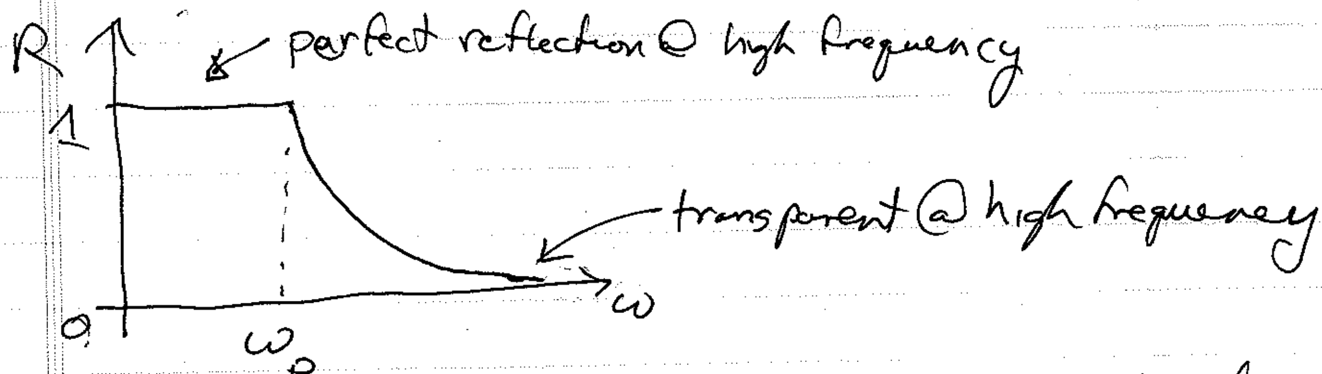
Index in plasma: $n(\omega) = \frac{c}{v_p(\omega)} = \frac{ck}{\omega}$

$$\Rightarrow n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

$$\therefore \boxed{R = \left| \frac{\sqrt{1 - \frac{\omega_p^2}{\omega^2}} - 1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}} - 1} \right|^2}$$

When $\omega < \omega_p$ $n(\omega) = i\sqrt{\frac{\omega_p^2}{\omega^2} - 1}$ (pure imaginary)

$\Rightarrow \boxed{R = 1}$ Perfect reflection for evanescent wave



This effect is seen in the "ionosphere" where AM radio is reflected differently in sunlight vs. evening.

Problem 3: Drude model of a conductor

Under the Drude model, the conduction electrons are free particles, but undergo damping due to collisions with defects in the crystal.

(a) We seek the conductivity according to Ohm's Law

$$\underset{\sim}{\vec{J}}(\omega) = \tilde{\sigma}(\omega)\,\underset{\sim}{\vec{E}}(\omega)$$

The current is defined in steady state due to the driving electric field

$$\vec{J}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{J}}(\omega)\, e^{-i\omega t}\right)$$

But $\vec{J}(t) = \underset{\text{charge density}}{\rho}\,\vec{v}(t) = -e\underset{\substack{\text{number density}\\ \text{of electrons}}}{N}\,\vec{v}(t)$

Thus we seek the steady-state velocity

$$\vec{v}(t) = \mathrm{Re}\left(\vec{v}(\omega)\, e^{-i\omega t}\right)$$

The eq of motion follows from Newton's Law

$$m_e \frac{d\vec{v}}{dt} = -\underset{\text{damping}}{m_e\gamma}\,\vec{v} - \underset{\text{force of field}}{e\vec{E}(t)}$$

Under the assumption of harmonic driving

$$\vec{E}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{E}}(\omega)\, e^{-i\omega t}\right)$$

$$\Rightarrow -i\omega m_e \underset{\sim}{\vec{v}}(\omega) = -m\gamma\,\underset{\sim}{\vec{v}}(\omega) - e\underset{\sim}{\vec{E}}(\omega)$$

$$\Rightarrow \quad \tilde{v}(\omega) = \left[\frac{\frac{-e}{m\gamma}}{1 - i\frac{\omega}{\gamma}}\right] \tilde{E}(\omega)$$

Thus $$\tilde{\vec{J}}(\omega) = \left[\frac{\frac{Ne^2}{m\gamma}}{1 - i\frac{\omega}{\gamma}}\right] \vec{E}(\omega)$$

$$\Rightarrow \boxed{\tilde{\sigma}(\omega) = \frac{\left(\frac{\omega_p^2}{\gamma}\epsilon_0\right)}{1 - i\frac{\omega}{\gamma}} \leftarrow \sigma_{DC}, \quad \text{where } \omega_p^2 = \frac{Ne^2}{\epsilon_0 m_e}}$$

(b) Given, silver $\begin{cases} \sigma_{DC} = 6.17 \times 10^{17} / (\text{ohm} \cdot m) \\ \gamma = 2.5 \times 10^{13}\, s^{-1} \end{cases}$

We found in class the complex index of refraction

$$\tilde{n}(\omega) = \sqrt{1 + i\frac{\tilde{\sigma}(\omega)}{\omega\epsilon_0}} \quad \left(\text{taking } \begin{matrix} \epsilon = \epsilon_0 \\ \mu = \mu_0 \end{matrix}\right)$$

At a radio frequency $\nu \sim 1$ MHz

$\omega = 6.2 \times 10^7\, s^{-1} \ll \gamma$

and $\sigma_{DC} \gg \omega\epsilon_0$ (Good conductor)

$$\Rightarrow \tilde{\sigma}(\omega) \simeq \sigma_{DC} \qquad \tilde{n}(\omega) \approx \sqrt{i\frac{\sigma_{DC}}{\omega\epsilon_0}}$$

$$= \sqrt{\frac{\sigma_{DC}}{2\omega\epsilon_0}}\,(1+i)$$

$$\Rightarrow \boxed{\underset{\substack{\text{real part}\\ \downarrow}}{} n_R(\omega) \approx \underset{}{} n_I(\omega) \approx \sqrt{\frac{\sigma_{DC}}{2\omega\epsilon_0}}}$$

real part $\downarrow$ $n_R(\omega)$; Imag part $\Downarrow$ $n_I(\omega)$

Skin depth = distance for intensity to decay to $\frac{1}{e}$

$$d = \frac{1}{2k_I} = \frac{1}{2\frac{\omega}{c}n_I(\omega)} \approx \frac{1}{\sqrt{2\omega\mu_0\sigma_{DC}}}$$

Plugging in with $\frac{\omega}{2\pi} = 1\ \text{MHz}$

$$\Rightarrow \quad d \approx 2\times10^{-9}\,m = 2\ nm$$