Physics 406: Electricity and Magnetism II

Problem Set #8: DUE Friday, Nov. 2, 2012

Reminder: Exam#2, Nov. 8, in class.

Problem 1 Anomalous dispersion in a dielectric (10 points)

The group velocity is defined in an absorptive dispersive medium is defined as $v_g = \dfrac{1}{\dfrac{dk_R(\omega)}{d\omega}}$, where $k_R(\omega)$ is the real part of the wave number.

(a) Show that $v_g(\omega) = \dfrac{c}{n_R(\omega) + \omega\left(dn_R/d\omega\right)}$,

(b) For a single resonance in the Lorentz model, we found,

$$n_R(\omega) = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}, \text{ or near resonance, } n_R(\omega) \approx 1 + A\left(\frac{-\Delta}{\Delta^2 + \Gamma^2/4}\right)$$

$$\text{where } \Delta = \omega - \omega_0 \text{ and } A = \frac{\omega_p^2}{2\omega_0}.$$

Sketch the phase velocity as a function of Δ, comment.

(c) Show that in the region of anomalous dispersion $dn_R/d\omega$<0 even the group velocity can be greater than c!! What is the condition on the index of refraction? (Give a general analytic expression. You need not evaluate it explicitly for the case under consideration).

The physical meaning of this is subtle. Rest assured though, this does not mean that any signal can be sent faster than the speed of light

Problem 2. Chromatic dispersion in an optical fiber. (10 Points)

Dispersion and the resulting spreading of a wave packet is a serious problem in optical communications, where pulses of light are sent through optical fibers over very long distances. Optical fibers are basically wires of silica glass. The minimum absorption (due to scattering from imperfections and other loss mechanisms) is at a wavelength of λ=1.55 μm. (The power) attenuation coefficient at this frequency is found to be $2\beta=2\text{x}10^{-5}$ cm^{-1}, and the dispersion coefficient $\dfrac{d^2k}{d\omega^2}$ is about -25 ps^2/km. (ps= picosecond)

(a) Estimate the characteristic distance of spreading of a 20 ps pulse at this wavelength. How does it compare with the attenuation length?

(b) One way to send a digital signal in an optical fiber is divide up time into "windows" of duration T. If a pulse appears in the window, the bit is a "1" - no pulse, the bit is a "0".

The rate of data transmission (bits/second) is then $1/T$. The window is taken in order to avoid overlap. It can be shown that if we can tolerate a bit rate error no greater than 10^{-9}, then the time window must be 2 times the pulse duration. Of course the name of the game is to maximize the bit-rate over the longest possible distances. However, very short pulses have broad spectra, and thus spread very rapidly.

What is the maximum distance one could send 20 ps pulses at 10 Gbit/sec (in a 100 ps time window) before dispersion makes the bit error-rate intolerable. **Is absorption a problem?**

Problem 1: Propagation and Dispersion of a Gaussian Pulse (15 Points)

(a) Show that the Fourier Transform of a Gaussian is a Gaussian, i.e., given $f(t) = Ae^{-\frac{t^2}{2(\Delta t)^2}}$ (Gaussian of temporal duration Δt), show $\tilde{f}(\omega) = A\sqrt{2\pi\Delta t^2}e^{-\omega^2\Delta t^2/2}$.

Hint: Complete the square in the exponent and use a Gaussian integral. $\int_{-\infty}^{\infty} dt \exp[-\alpha(t-\beta)^2] = \sqrt{\pi/\alpha}$, where α,β can be complex, and Re(α)>0.

What us the characteristic spectral bandwidth, Δω?

(b) Consider a quasimonochromatic pulse $E(t) = E_0 e^{-t^2/(2\Delta t_0^2)}\cos(\omega_0 t)$, where $\Delta t >> 2\pi/\omega_0$.

Show that the Fourier transform on the pulse is

$$\tilde{E}(\omega) = \sqrt{\frac{\pi\Delta t_0^2}{2}} E_0\left(e^{-(\omega-\omega_0)^2\Delta t_0^2/2} + e^{-(\omega+\omega_0)^2\Delta t_0^2/2}\right).$$

Sketch this function.

(c) Suppose this signal is incident on a medium with $k(\omega)$. Including group-velocity dispersion, i.e. keeping terms up to second order, $k(\omega) = k_0 + (\omega-\omega_0)k_0' + \frac{1}{2}(\omega-\omega_0)^2 k_0''$, show that the wave inside the medium is

$$E(z,t) = \mathrm{Re}\left[E_0 \frac{\Delta t_0}{\sqrt{\Delta t_0^2 - ik_0''z}} e^{-\frac{(t-z/v_g)^2}{2\left(\Delta t_0^2 - ik_0''z\right)}} e^{i(k_0 z - \omega_0 t)}\right],$$

where $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$ is the group velocity.

(d) Show that the intensity of the pulse inside the medium is

$$I(z,t) = I_0 \frac{\Delta t_0}{\Delta t_z} \exp\left[-\frac{(t-z/v_g)^2}{\Delta t_z^2}\right], \text{ where, } I_0 = \sqrt{\frac{\varepsilon}{\mu_0}}\frac{|E_0|^2}{2} \text{ and } \Delta t_z^2 = \Delta t_0^2 + \frac{(k_0''z)^2}{\Delta t_0^2}.$$

Sketch the intensity as a function of t for different z.