Physics 406

Problem Set #8: Solutions

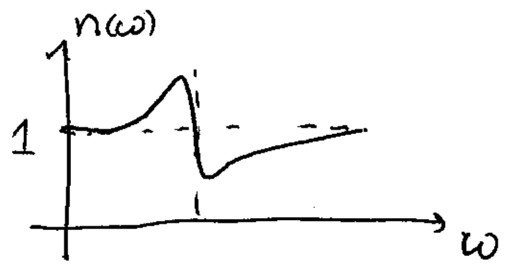
Problem 1: Anomalous dispersion

(a) The dispersion relation $k(\omega) = \frac{\omega}{c} n_R(\omega)$ ← real index

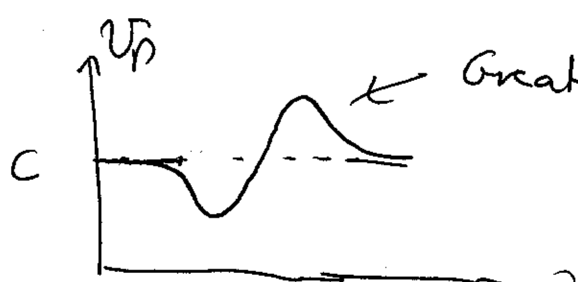
$$\Rightarrow \frac{dk}{d\omega} = \frac{n_R(\omega)}{c} + \frac{\omega}{c}\frac{dn_R}{d\omega}$$

$\Rightarrow$ Group velocity $\boxed{v_g(\omega) = \frac{c}{n_R(\omega) + \omega \frac{dn_R}{d\omega}}}$

(b) Single resonance Lorentz Model:

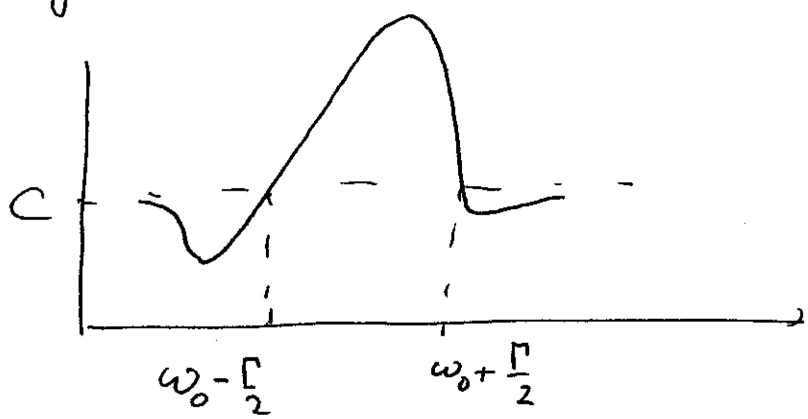


Phase velocity $v_p(\omega) = \frac{c}{n_R(\omega)}$



← Greater than c! No worries, the phase velocity does not describe the speed at which a signal travels.

C) For the case of the dielectric, a sketch of $v_g(\omega)$ in the region of resonance



Condition for $v_g > 0$

$$\boxed{n_R + \omega \frac{dn_R}{d\omega} < 1}$$

Outside the region of resonance v_g is less than c, But right inside the absorption line it isn't!

What does this mean? In all the classic text books, e.g. Jackson's "Classical Electrodynamics" and Born and Wolf "Optics", it is stated that when $v_g > c$ it is not a useful quantity. It should be noted that in the region of anamolous dispersion $n_R(\omega)$ is a rapidly varying function of ω, and thus the Taylor series expansion may not be valid. However there are situations where the Taylor series still converges and $v_g > c$. The physical meaning of this is still under dispute.

<u>See for reference:</u>

- L. Brillouin, "Wave Propagation and Group Velocity" (Academic Press, New York, 1952).
- R.Y. Chiao et al., in Proceedings of the conference, "Fundamentals Problems in Quantum Theory" (Ann. N.Y. Acad. Sci.) 1994
- E.L. Bolda et al., Phys Rev. A <u>48</u> 3890 (1993).
  ibid, Phys Rev A, <u>49</u> 2071 (1994).

Problem 2 Chromatic Dispersion in Optical Fibers

(a) 20 ps pulse at $\lambda = 1.55\,\mu m$

Characteristic distance of spreading

$$z_{spread} = \frac{\Delta t(0)^2}{\left|\frac{d^2k}{d\omega^2}\right|} = \frac{400\,ps^2}{25\,ps^2/km} = \boxed{16\ km}$$

The power is attenuated by $P(z) = e^{-2\beta z} P(0)$

$\Rightarrow$ The characteristic attenuation length

$$z_{attenuation} = \frac{1}{2\beta} = \frac{1}{2\times10^{-5}\,cm^{-1}} = 5\times10^4\,cm = \boxed{0.5\,km}$$

Yes attenuation is a problem over distances of a few kilometers. This can be overcome through the use of light amplifiers (laser) directly inside the fiber!

Note, however, how incredibly transparent glass is at this frequency in any event

(b) 20 ps pulse at 10 Gbits/sec = Bit rate

Time window $T = \frac{1}{\text{Bit rate}} = \frac{1}{10\times10^9\,sec^{-1}} = 10^{-10}\,sec$
$= 100\,ps$

For a bit-rate-error $< 10^{-9}$, we require pulse duration

$$\Delta t < T/2 = 50\,ps$$

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As a functions of propagation distance, the pulse width increases according to

$$\Delta t(z) = \sqrt{(\Delta t(0))^2 + \left(\frac{k'' z}{\Delta t(0)}\right)^2}$$

$$= \sqrt{1 + \left(\frac{z}{z_{spread}}\right)^2}\ \Delta t(0)$$

Maximum propagation distance: z_{max}

$$\Delta t(z_{max}) = \sqrt{1 + \left(\frac{z_{max}}{z_{spread}}\right)^2}\ \Delta t(0) < \frac{T}{2}$$

$$\Rightarrow z_{max} = \sqrt{\left(\frac{T}{2\,\Delta t(0)}\right)^2 - 1}\ z_{spread}$$

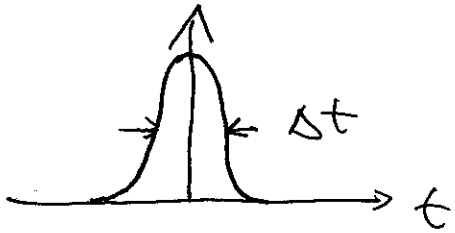
$$\Rightarrow z_{max} = \sqrt{\left(\frac{50}{25}\right)^2 - 1}\ z_{spread} \simeq 4.6\ z_{spread}$$

$$\boxed{z_{max} \approx 73.6\ \text{km}}$$

This is not so bad for some local data transmissions, but it's deadly for long distance communications. One solution to this problem is to make use of some nonlinear effects in the fiber which can counteract the spreading. These stable, non-spreading, pulses are known as "solitons" and appear in a variety of nonlinear dynamical systems. Other solutions include engineering the fiber to reduce dispersion near the low loss window, and cleverer schemes for coding the data. These are hot topics in optical communications

Problem 3: Gaussian Pulse

(a) Gaussian function $f(t) = A e^{-\frac{t^2}{2(\Delta t)^2}}$



Fourier transform

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, e^{+i\omega t} f(t) = A \int_{-\infty}^{\infty} dt\, \exp\left\{ \frac{-t^2}{2(\Delta t)^2} + i\omega t \right\}$$

Aside: Complete the square in the exponent

$$\frac{-t^2}{2\Delta t^2} + i\omega t = -\frac{1}{2\Delta t^2}\left(t^2 - 2i\omega t\, \Delta t^2\right)$$

$$= -\frac{1}{2\Delta t^2}\left(t^2 - 2i\omega t\,\Delta t^2 + (i\omega \Delta t^2)^2\right) + \frac{(i\omega\Delta t^2)^2}{2\Delta t^2}$$

$$= -\frac{1}{2\Delta t^2}\left(t - i\omega\Delta t^2\right)^2 - \frac{1}{2}\omega^2 \Delta t^2$$

$$\Rightarrow \tilde{f}(\omega) = A e^{-\frac{1}{2}\omega^2\Delta t^2} \int_{-\infty}^{\infty} dt\, e^{-\frac{1}{2\Delta t^2}(t - i\omega\Delta t^2)^2}$$

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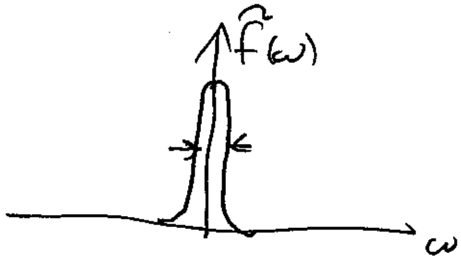
Now use the Gaussian integral formula:

$$\int_{-\infty}^{\infty} dt\, e^{-\alpha(t-\beta)^2} = \sqrt{\frac{\pi}{\alpha}} \quad \text{if real part of } \alpha > 0$$

Here $\alpha = \frac{1}{2\Delta t^2}$ $\beta = i\omega\Delta t^2$

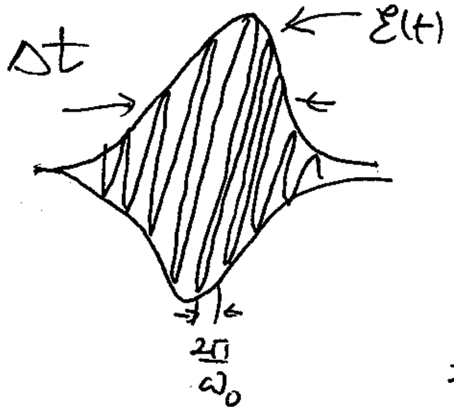
$$\Rightarrow \boxed{\tilde{f}(\omega) = \sqrt{2\pi\Delta t^2}\, A\, e^{-\frac{\omega(\Delta t)^2}{2}}}$$

Gaussian with $\sqrt{\text{variance}}$ $\boxed{\Delta\omega = \frac{1}{\Delta t}}$



<u>Lesson</u>: Fourier ~~the~~ transform of a Gaussian is a Gaussian whose spectral width is inversely proportion the signal duration

(b) Now we have a quasi-monochromatic Gaussian pulse



Not to scale

$$E(t) = \mathcal{E}(t)\cos\omega_0 t$$

$$\mathcal{E}(t) = E_0 e^{-\frac{t^2}{2\Delta t_0^2}}$$

$$\Rightarrow E(t) = \mathrm{Re}\left(\mathcal{E}(t)\, e^{-i\omega_0 t}\right)$$

$$= \frac{1}{2}\mathcal{E}(t)\, e^{-i\omega_0 t} + \frac{1}{2}\mathcal{E}\, e^{+i\omega_0 t}$$

(Having used $\cos\omega_0 t = \frac{e^{-i\omega_0 t} + e^{+i\omega_0 t}}{2}$, and here $\mathcal{E}(t)$ is real)

$$\therefore \tilde{E}(\omega) = \int_{-\infty}^{\infty} dt\, e^{+i\omega t} E(t) = \frac{1}{2}\int_{-\infty}^{\infty} dt\, \mathcal{E}(t)\, e^{i(\omega-\omega_0)t}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} dt\, \mathcal{E}(t)\, e^{i(\omega+\omega_0)t}$$

$$= \frac{1}{2}\tilde{\mathcal{E}}(\omega-\omega_0) + \frac{1}{2}\tilde{\mathcal{E}}(\omega+\omega_0)$$

Fourier transform of Gaussian envelope, found in part (a)

$$\Rightarrow \tilde{E}(\omega) = \frac{1}{2}\left(\sqrt{2\pi\Delta t^2}\, E_0\right)\left(e^{-\frac{(\omega-\omega_0)^2}{2}\Delta t_0^2} + e^{-\frac{(\omega+\omega_0)^2}{2}\Delta t_0^2}\right)$$

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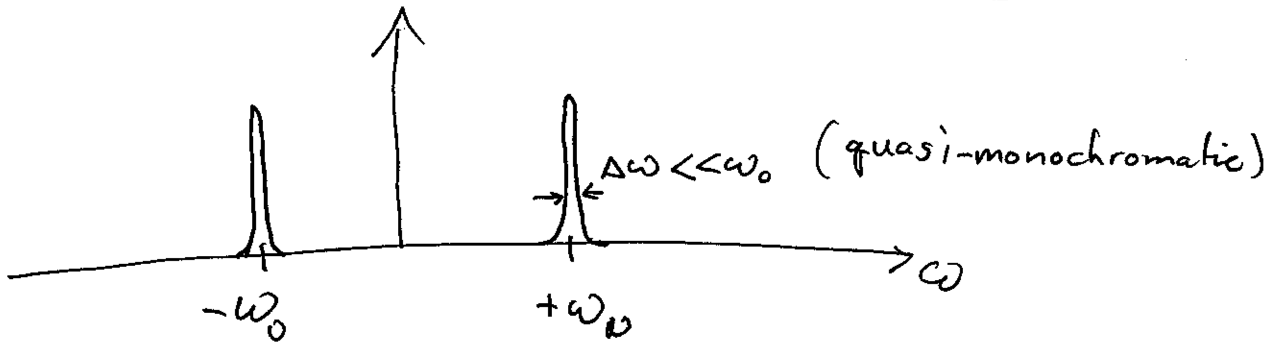
Thus:

$$\boxed{\tilde{E}(\omega) = \sqrt{\frac{\pi \Delta t_0^2}{2}} E_0 \left(e^{-(\omega+\omega_0)^2 \frac{\Delta t^2}{2}} + e^{-(\omega-\omega_0)^2 \frac{\Delta t^2}{2}} \right)}$$



(c) We showed in Lecture:

Given $E(t) = \mathcal{E}(t)\cos\omega_0 t$ @ $z=0$

with $\mathcal{E}(t)$ a slowly varying envelope (quasi-monochromatic)

$$E(z,t) \simeq \mathrm{Re}\int_{-\infty}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(\Omega)\, e^{i(k(\omega_0+\Omega)z - (\omega_0+\Omega)t)}$$

$\uparrow$ Fourier transform of envelope

Expanding $k(\omega_0+\Omega)$ about $\Omega=0$ to second order

$$\Rightarrow k(\omega_0+\Omega) = k(\omega_0) + \Omega \left.\frac{dk}{d\omega}\right|_{\omega_0} + \frac{1}{2}\Omega^2 \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}$$

$$\equiv k_0 + \frac{\Omega}{v_g} + \frac{k_0''}{2}\Omega^2$$

where $k_0 \equiv k(\omega_0)$, $\frac{1}{v_g} \equiv \left.\frac{dk}{d\omega}\right|_{\omega_0}$, $k_0'' \equiv \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}$

$$\Rightarrow E(z,t) = \mathrm{Re}\left[\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\,\tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega(t - \frac{z}{v_g})}\; e^{\frac{i\Omega^2 k_0'' z}{2}}\; e^{i(k_0 z - \omega_0 t)}\right]$$

Now plug in ~~...~~ for $\tilde{\mathcal{E}}(\Omega) = E_0\sqrt{2\pi\Delta t_0^2}\; e^{-\frac{\Omega^2\Delta t_0^2}{2}}$

$$\Rightarrow E(z,t) = \mathrm{Re}\left[\left(\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\, E_0\sqrt{2\pi\Delta t_0^2}\;\exp\left[-\frac{\Omega^2\Delta t_0^2}{2} + \frac{i\Omega^2 k_0'' z}{2} - i\Omega\left(t - \frac{z}{v_g}\right)\right]\right) e^{i(k_0 z - \omega_0 t)}\right]$$

complete the square in Ω

Aside

$$-\frac{1}{2}\left(\Delta t_0^2 - i k_0'' z\right)\left[\Omega^2 - \frac{2i\Omega\left(t - \frac{z}{v_g}\right)}{\Delta t_0^2 - i k_0'' z}\right]$$

$$= -\frac{1}{2}\left(\Delta t_0^2 - i k_0'' z\right)\left(\Omega - \frac{i\left(t - \frac{z}{v_g}\right)}{\cancel{2}(\Delta t^2 - i k_0'' z)}\right)^2 - \frac{1}{2}\frac{\left(t - \frac{z}{v_g}\right)^2}{(\Delta t^2 - i k_0'' z)^2}$$

$$\therefore E(z,t) = \mathrm{Re}\left[\left(\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\, E_0\sqrt{2\pi\Delta t_0^2}\; e^{-\alpha(\Omega - \beta)^2}\right) e^{-\frac{\left(t - \frac{z}{v_g}\right)^2}{2(\Delta t_0^2 - i k_0'' z)^2}}\; e^{i(k_0 z - \omega_0 t)}\right]$$

where $\alpha = \frac{\Delta t_0^2 - i k_0'' z}{2}$ $\quad \mathrm{Re}(\alpha) > 0$

$\beta = \frac{i(t - z/v_g)}{\cancel{2}(\Delta t_0^2 - i k_0 z)}$

$$\therefore\ E(z,t) = Re\left[\left(\underbrace{\frac{1}{2\pi}\sqrt{2\pi\Delta t_0^2}\sqrt{\frac{\pi}{\alpha}}}_{E_0}\right) e^{-\frac{(t-\frac{z}{v_g})^2}{2(\Delta t_0^2 - ik_0'' z)}}\ e^{i(k_0 z - \omega_0 t)}\right]$$

plug in for α

$$\Rightarrow \boxed{E(z,t) = Re\left[E_0 \frac{\Delta t_0}{\sqrt{\Delta t_0^2 - ik_0'' z}}\ e^{-\frac{(t-\frac{z}{v_g})^2}{2(\Delta t_0^2 - ik_0'' z)}}\ e^{i(k_0 z - \omega_0 t)}\right]}$$

(d) Intensity: $I = \frac{1}{2}\tilde{E}\tilde{H}^* = \frac{|\tilde{E}|^2}{2Z}$

$Z = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon}}$ (impedance of wave)

$\tilde{E}$ is the complex amplitude

$$E(z,t) = Re\left(\tilde{E}(z,t)\, e^{-i\omega_0 t}\right)$$

$$\Rightarrow \tilde{E}(z,t) = E_0 \frac{\Delta t_0}{\sqrt{\Delta t_0^2 - ik_0'' z}}\ e^{-\frac{(t-\frac{z}{v_g})^2}{2(\Delta t_0^2 - ik_0'' z)}}\ e^{ik_0 z}$$

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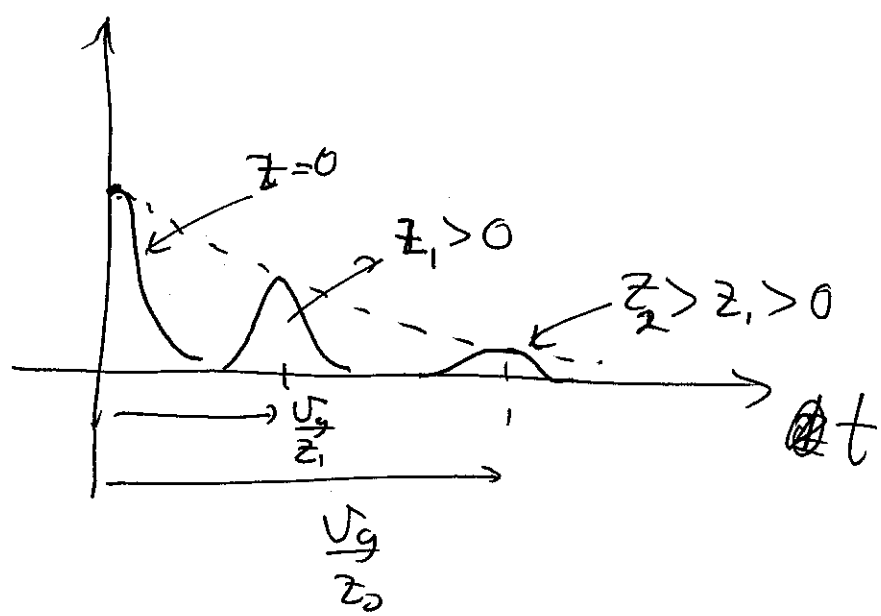
$$|\tilde{E}(z,t)|^2 = \tilde{E}^*(z,t)\,\tilde{E}(z,t)$$

$$= E_0^2 \frac{\Delta t_0^2}{\sqrt{\Delta t_0^4 + (k_0'' z)^2}}\, e^{-\frac{(t-\frac{z}{v_g})^2 \Delta t_0^2}{(\Delta t_0^4 + (k_0'' z)^2)}}$$

$$= E_0^2 \frac{\Delta t_0}{\sqrt{\Delta t_0^2 + \left(\frac{k_0'' z}{\Delta t_0}\right)^2}} \exp\left[\frac{-(t-\frac{z}{v_g})^2}{\Delta t_0^2 + \left(\frac{k_0'' z}{\Delta t_0}\right)^2}\right]$$

$$\Rightarrow I(z,t) = I_0 \frac{\Delta t_0}{\Delta t_z}\, e^{-\frac{(t-\frac{z}{v_g})^2}{\Delta t_z^2}}$$

where $\Delta t_z = \sqrt{\Delta t_0^2 + \left(\frac{k_0'' z}{\Delta t_0}\right)^2}$



Propagating, spreading pulse