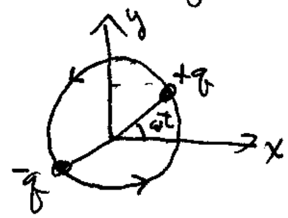
Physics 406
Problem Set #9 Solutions

(1) Radiation by a rotating electric dipole



$$\vec{p}(t) = p_0 \cos(\omega t)\,\hat{x} + p_0 \sin(\omega t)\,\hat{y}$$

Using the principle of superposition, we can add the dipole radiation associated with the dipole along x to that for the dipole along y.

For a monochromatic oscillating dipole, we have the complex amplitude for the electric field

$$\underset{\sim}{\vec{E}} = +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \underset{\sim}{\vec{p}}_{0\perp} \quad \text{where} \quad \vec{p}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{p}_0}\, e^{-i\omega t}\right)$$

$\Leftarrow$ component $\perp$ to direction of observation

$$= +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \left(\underset{\sim}{\vec{p}_0} - \hat{r}\,(\hat{r}\cdot\underset{\sim}{\vec{p}_0})\right)$$

In this case: $\vec{p}_0(t) = \mathrm{Re}\left(p_0 e^{-i\omega t}\,\hat{x} + i p_0 e^{-i\omega t}\,\hat{y}\right)$

90° out of phase

$$\Rightarrow \underset{\sim}{\vec{E}} = +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} p_0 \left\{ \left(\hat{x} - \hat{r}(\hat{r}\cdot\hat{x})\right) + i\left(\hat{y} - \hat{y}(\hat{r}\cdot\hat{y})\right)\right)$$

Problem 1 continued

From the inside back cover of Griffiths, we can express the Cartesian basis vectors in terms of those in spherical coordinates (which depends upon the point of observation)

$$\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\Rightarrow \hat{x} - \hat{r}(\hat{x}\cdot\hat{r}) = \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\Rightarrow \hat{y} - \hat{r}(\hat{y}\cdot\hat{r}) = \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

Plug into the expression for the electric field complex amplitude

$$\Rightarrow \tilde{\vec{E}} = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{e^{ikr}}{r}\left((\cos\theta\cos\phi + i\cos\theta\sin\phi)\,\hat{\theta} + (-\sin\phi + i\cos\phi)\,\hat{\phi} \right)$$

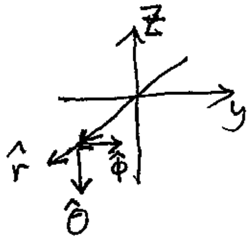
$$= \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{e^{ikr}}{r}\left(\cos\theta\,\underbrace{(\cos\phi + i\sin\phi)}_{=e^{i\phi}}\,\hat{\theta} + i\,\underbrace{(\cos\phi + i\sin\phi)}_{=e^{i\phi}}\,\hat{\phi} \right)$$

$$\Rightarrow \boxed{\tilde{\vec{E}} = \frac{1}{4\pi\epsilon_0} k^2 p_0 \left(\cos\theta\,\hat{\theta} + i\,\hat{\phi}\right) \frac{e^{i(kr+\phi)}}{r}}$$

The phase of the wave $kr + \phi = \omega\frac{r}{c} + \phi$, where (r, ϕ) are the coordinates of the observer (in the radiation zone). This makes sense: The ~~first~~ first term is the retarded time effect. The phase of oscillation should depend on the phi coordinate because the dipole in rotating in the ϕ-direction. Thus observers at different ϕ will see the wave at a different phase of the oscillation.

(b) Problem 1 continued

On the x-axis: $\theta = \frac{\pi}{2}$, $\phi = 0$, $y = z = 0 \Rightarrow r = x$



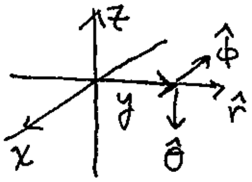
$\hat{r} = \hat{x}$, $\hat{\theta} = -\hat{z}$, $\hat{\phi} = \hat{y}$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \,\mathrm{Re}\left(i\, \hat{y}\, \frac{e^{i(kx-\omega t)}}{x}\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{\sin(\omega t - kx)}{x}\, \hat{y}}$$

Linear polarization along $\hat{y}$

On the y-axis: $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, $x = z = 0 \Rightarrow r = y$



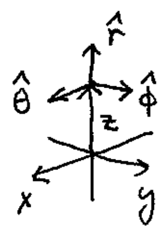
$\hat{r} = \hat{y}$, $\hat{\theta} = -\hat{z}$, $\hat{\phi} = -\hat{x}$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \,\mathrm{Re}\left(-i\, \hat{x}\, \frac{e^{i(ky-\omega t)}}{y}\, \underbrace{e^{i\pi/2}}_{=i}\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{\cos(\omega t - ky)}{y}\, \hat{x}}$$

Linear polarization along $\hat{x}$

On the z-axis $\theta = 0$, $\phi = 0$, $x = y = 0 \Rightarrow r = z$



$\Rightarrow \hat{r} = \hat{z}$, $\hat{\theta} = \hat{x}$, $\hat{\phi} = \hat{y}$

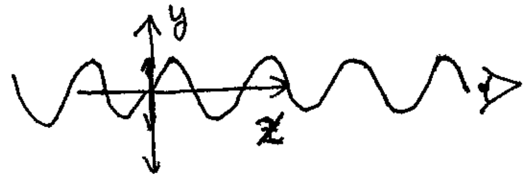
$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \,\mathrm{Re}\left\{(\hat{x} + i\hat{y})\, \frac{e^{ikz}}{z}\, e^{-i\omega t}\right\}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \left(\frac{\cos(\omega t - kz)\,\hat{x} + \sin(\omega t - kz)\,\hat{y}}{z}\right)}$$

Right hand circular polarization

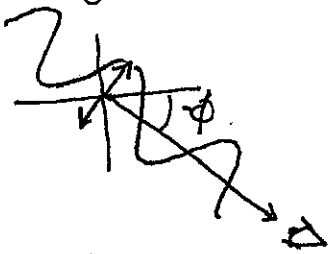
Comment on wave polarization

An observer on the x-axis sees only the projection of the rotating dipole along the y-axis, and vice-versa for observers on the y-axis



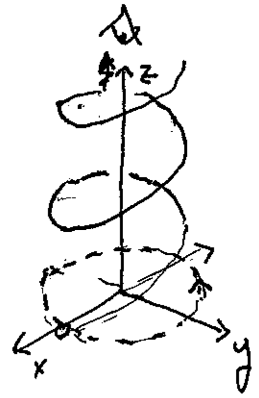
Generally, anywhere in the x-y plane

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \cos(\omega t - k\sqrt{x^2+y^2} + \phi)\ \hat{\phi}$$



Linear polarization along $\hat{\phi}$, phase of oscillation depends on ϕ

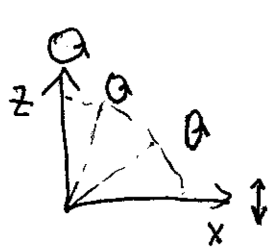
On the z-axis



Traveling helix (right hand cork screw)

$\Rightarrow$ R.H. circular polarization

For an arbitrary point not along one of the axes the polarization is <u>elliptical</u> since the observer does not see an equal projection of oscillations along the two directions $\perp$ to the direction of observation



(c) The intensity

$$\langle S \rangle = c\,\frac{\epsilon_0}{2}\,\underset{\sim}{\vec{E}}^{*}\cdot\underset{\sim}{\vec{E}} = c\,\frac{\epsilon_0}{2}\left(\frac{1}{16\pi^2\epsilon_0^2}\,k^4 p_0^2\,(\cos^2\theta+1)\,\frac{1}{r^2}\right)$$

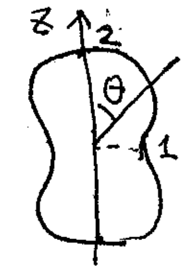
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4 p_0^2}{8\pi}\right)\frac{1}{r^2}\,(1+\cos^2\theta)$$

The power radiated into the element of area subtended by the solid angle $d\Omega$: $dP = \langle S\rangle dA = \langle S\rangle r^2 d\Omega$

$\Rightarrow$ differential $\frac{\text{Power}}{\text{solid angle}}$ $\boxed{\frac{dP}{d\Omega} = \langle S\rangle r^2 = \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)(1+\cos^2\theta)}$

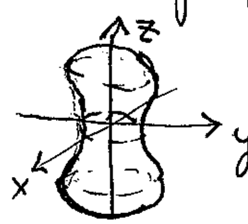
Polar plot: At $\theta=0$ $\frac{dP}{d\Omega}$ twice that at $\theta=\frac{\pi}{2}$. $\cos^2\theta$ increases as θ changes from $\frac{\pi}{2}$

$\frac{dP}{d\Omega}$



Units of $\frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)$

Azimuthally symmetric



(d) Total radiated power into all 4π steradians:

$$P_{total} = \int \frac{dP}{d\Omega}\,d\Omega = \int \frac{dP}{d\Omega}\,2\pi\sin\theta\,d\theta = \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)2\pi\int_0^{\pi}(1+\cos^2\theta)\times\sin\theta\,d\theta$$

Let $\mu=\cos\theta \Rightarrow d\mu = -\sin\theta\,d\theta$

$$\therefore \int_0^{\pi}(1+\cos^2\theta)\sin\theta\,d\theta = \int_{-1}^{1}(1+\mu^2)\,d\mu = \left(\mu+\frac{\mu^3}{3}\right)\Big|_{-1}^{1} = \frac{8}{3}$$

$$\therefore \boxed{P_{total} = \frac{1}{4\pi\epsilon_0}\left(\frac{2}{3}ck^4p_0^2\right)}$$

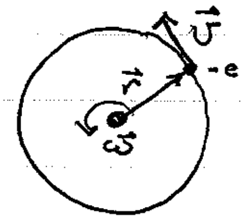
Factor of two because we have the superposition of <u>two</u> dipoles

(No interference term)
: orthogonal fields

(2) Decay of a classical atom

Hydrogen: Electron bound to a proton

(Circular orbit)



Instantaneous energy: E = Kinetic + potential

$$\Rightarrow E = \frac{1}{2} m v^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Instantaneous force: $|\vec{F}_{centripetal}| = |\vec{F}_{coulomb}|$

$$\Rightarrow \frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

$$\therefore E = \frac{-1}{4\pi\epsilon_0} \frac{e^2}{2r}$$ (Also follows from Virial Theorem: $\langle \text{Kinetic} \rangle = -\frac{1}{2} \langle rF \rangle$)

As the electron orbits the nucleus it will radiate electromagnetic energy since it is constantly accellerating.
From problem 1, $P_{rad} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} c k^4 p_0^2$, $p_0 = er$

$$\Rightarrow \frac{dE}{dt} = -\frac{d}{dt}\left(\frac{1}{4\pi\epsilon_0} \frac{e^2}{2r}\right) = +\frac{1}{4\pi\epsilon_0} \frac{e^2}{2r^2} \frac{dr}{dt} = \underset{\substack{\uparrow \\ \text{decrease} \\ \text{in energy}}}{-} P_{rad}$$

$$= -P_{rad} = \frac{-1}{4\pi\epsilon_0} \frac{2}{3} \frac{\omega^4}{c^3} e^2 r^2$$

$$\Rightarrow \frac{dr}{dt} = -\frac{4}{3} \frac{\omega^4}{c^3} r^4$$

The instaneous angular frequency $\omega = \frac{v}{r} \Rightarrow \omega = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r^3}}$

$$\therefore \frac{dr}{dt} = -\frac{4}{3} c \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 \frac{1}{r^2}$$

(Next Page)

Define the "classical energy radius"

$$r_{class} = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ meters}$$

(This is the radius such that the rest mass of the electron is attributed to the energy necessary to assemble the charge e on a sphere of radius r_{class}: $mc^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_{class}}$)

$$\Rightarrow 3r^2 \frac{dr}{dt} = -4c\, r_{class}^2 \quad \Rightarrow \quad \frac{d}{dt}(r^3) = -4c\, r_{class}^2$$

Solution: $r^3(t) = r^3(t=0) - 4ct\, r_{class}^2$

Take the initial radius to be the Bohr radius

$$r(t=0) = a_0 = .5\times 10^{-10} \text{ m} = 5 \times 10^{-11} \text{ m}$$

Decay time: $r(t_{decay}) = 0$ (electron crashes into nucleus)

$$\Rightarrow t_{decay} = \frac{a_0^3}{4c\, r_{class}^2} = \frac{(5\times 10^{-11} \text{ m})^3}{4(3\times 10^{10} \text{ m/s})(2.8\times 10^{-15} \text{ m})^2}$$

$$= 1.33 \times 10^{-13} \text{ s}$$

$$\Rightarrow \boxed{t_{decay} = 13.3 \text{ ps}}$$

So, classical atoms can't exist!

Quantum mechanics saves the day.