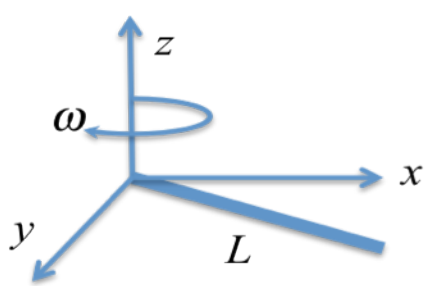
Physics 406: Electricity and Magnetism II

Problem Set #10: Due Fri. Nov 23, 2012

Problem 1. Radiation from a rotating rod (10 points)

A rod of length L and negligible cross-sectional area carries a total charge Q, uniformly distributed along its length. The rod is slowly rotated about one end with an angular frequency ω. The rod lies entirely in the x-y plane as it rotates, and $\omega L << c$.

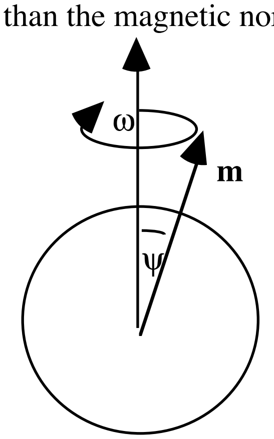


(a) What is the rate at which electric dipole energy is radiated by this system.

(b) Does this system radiate magnetic dipole radiation? Give your reasoning.

Problem 2. Magnetic Dipole Radiation (10 points)

Pulsars are thought to be rotating neutron stars, which have a strong magnetic moment, but which rotate about an axis different than the magnetic north pole (as does the earth)



Given a neutron star with typical radius of 10 km, a rotational period of 10^{-3} s, a surface magnetic field of 10^{8} Tesla, an a tilt of ψ=10 degrees. Calculate the energy radiated (as magnetic dipole radiation) per year and estimate the relative change per year (use the solar mass as an estimate of the star's mass). How long will such a pulsar "live"?

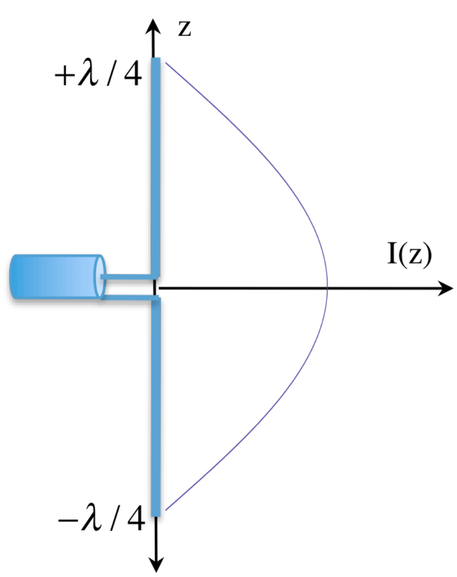
Problem 3: Antennas and Radiation Resistance (15 points)

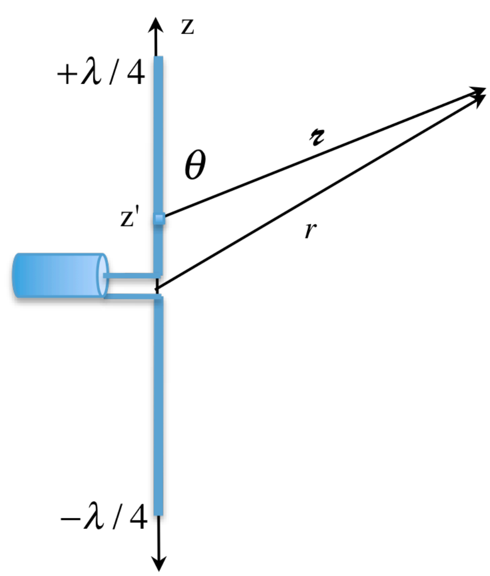
A system that radiates electromagnetic waves loses some of its energy. One can quantify this in terms of "radiation resistance", i.e., the equivalent resistance to the current that would cause the same loss of power (in a real resistor, the power is lost to heat through random collisions; here the power is lost to electromagnetic radiation).

(a) For the idealized model of a radiating dipole, with two points separated by distance a d and connected by thin wire (as in Griffiths and studied in class), show that the radiation resistance is

$$R_{rad} = 80\pi^2 \left(d/\lambda\right)^2 \text{ ohms}$$

The idealized radiating dipole in part (a) is not a real antenna because the current cannot be uniform across the whole length of the wire. A real, but simple antenna is a "center-fed", antenna, $I(z,t) = I(z)\cos\omega t$. Current flows, from the center out to the edge where it goes to zero, $I(z = \pm l/2) = 0$. We will take the length of the antenna to be ½ the wavelength (assume the current oscillate at ω), $l = \lambda/2$, so $I(z) = I_0 \cos(2\pi z/\lambda)$. Because the size of the system is not small compare to λ, we cannot take the whole antenna as single dipole, but we can break it up into infinitesimal pieces, each a radiating dipole, and then superpose all the components,





(b) Show that in the far-field, the complex amplitude of the radiated electric field is

$$\tilde{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{ik}{c}\frac{e^{ikr}}{r}\sin\theta \int_{-\lambda/4}^{\lambda/4} I(z') e^{-ikz'\cos\theta} dz'$$

(c) Do the integral to show that $\tilde{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{ik}{c}\frac{I_0 l\, e^{ikr}}{r}\left[\frac{2}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$.

(d) Show that the time-averaged power, radiated per solid angle is proportional to

$$\frac{dP_{rad}}{d\Omega} \propto \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}.$$

Plot (as a polar plot), and compare to an idealized "point dipole" radiation pattern.

(e) Extra credit: Numerically integrate to find the *total* time-averaged radiated power into all directions, and from this determine the radiation resistance. How does this compare to the answer in part (a) if we took $d = \lambda/2$? What should the impedance of the feeding coaxial cable be if all of the power is to be delivered to the antenna and not reflected.