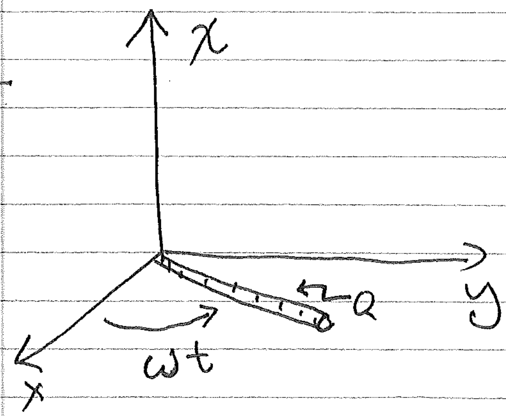
Physics 406: E&M II

Problem Set #10: Solutions

Prob. 1



A rotating rod has a line charge Q uniform distributed over its length. It rotates in the x-y plane at angular velocity ω, with $\omega \ll \frac{c}{L}$

Because the motion is slow, we can make the multipole expansion under the small parameter $\frac{\omega}{c} L = kL \ll 1$

$\Rightarrow$ The dominant power radiated is electric dipole radiation.

(a) We need to calculate the electric dipole moment, which will oscillate @ freq. ω

$$\vec{p}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{p}}\, e^{-i\omega t}\right)$$

By definition

$$\vec{p}(t) = \int_0^L ds\, \frac{Q}{L}\, s\, (\cos\omega t\, \hat{x} + \sin\omega t\, \hat{y})$$

$$= \underbrace{\frac{Q}{2L} s^2 \Big|_0^L}_{\frac{QL}{2}} \mathrm{Re}\left[(\hat{x} + i\hat{y})\, e^{-i\omega t}\right]$$

$$\Rightarrow \underset{\sim}{\vec{p}} = \frac{QL}{2}(\hat{x} + i\hat{y})$$

The time averaged electric dipole power radiated in electric dipole radiation is given by

$$\langle P^{E1}_{rad} \rangle = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{ck^4 |\vec{p}|^2}{3}$$

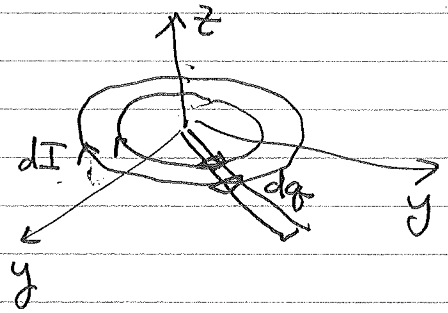
$$= \frac{1}{4\pi\epsilon_0} \frac{ck^4}{3} \left| \frac{QL}{2}(\hat{x}+i\hat{y}) \right|^2$$

$$\Rightarrow \boxed{\langle P^{E1}_{rad} \rangle = \frac{1}{4\pi\epsilon_0} \frac{ck^4}{6} (QL)^2}$$

(b) The power radiated as magnetic dipole radiation

$$P_{rad} \propto |\ddot{m}|^2$$

The key question, thus, is whether there is a magnetic dipole moment that has a ~~finite~~ nonzero second derivative.

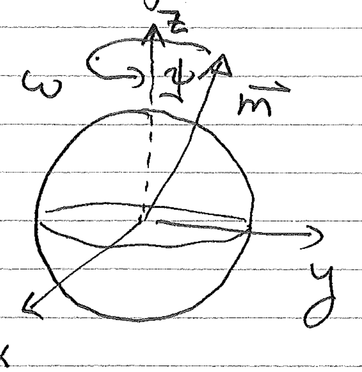


If we break up the rod into elements dq, each is rotating around the z-axis at constant speed ωr.

Thus, each current loop has ~~constant~~ constant current $\Rightarrow$ The magnetic dipole is constant

$$\boxed{\text{Thus, there is no magnetic dipole radiation}}$$

Problem 2: Magnetic Dipole Radiation



Rotating neutron star = Pulsar

$$\vec{m}(t) = |\vec{m}| \Big[\underbrace{\sin\psi \left(\cos\omega t \, \hat{x} + \sin\omega t \, \hat{y} \right)}_{\text{rotating component}} + \cos\psi \, \hat{z} \Big]$$

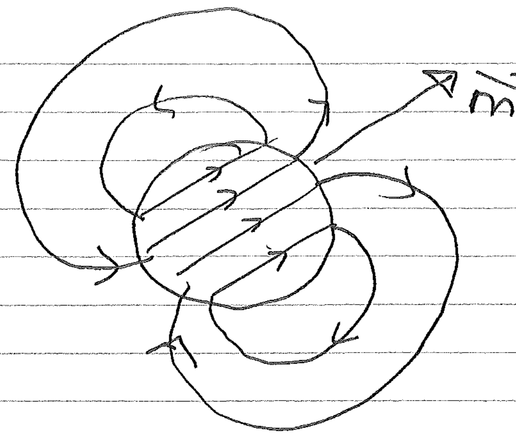
The rotating component will radiate magnetic dipole radiation at an average rate

$$\langle P_{rad} \rangle = \frac{1}{4\pi\epsilon_0} \frac{k^4 |\underset{\sim}{m}_{rot}|^2}{3c}$$

Here $\underset{\sim}{\vec{m}}_{rot} = |\vec{m}| \sin\psi \, (\hat{x} + i\hat{y})$

$$\Rightarrow \langle P_{rad} \rangle = \frac{1}{4\pi\epsilon_0} \frac{2 k^4 \sin^2\psi}{3c} |\vec{m}|^2$$

But, what is $|\vec{m}|$? We are given the magnetic field at the surface of the star. We can this of the neutron star as a uniformly magnetized sphere. In the near field, We know the field is the same as the (quasi) static $\vec{B}$-field



(Near) Field of a uniformly magnetized sphere

From Griffiths, Example 6.1, the magnetic field inside the sphere (up to the radius) is uniform, with the value

$$\vec{B} = \frac{2}{3}\mu_0 \vec{M}$$

$$\vec{M} = \frac{\vec{m}}{\frac{4}{3}\pi R^3} = \text{magnetization}$$

$$\Rightarrow \vec{m} = \frac{4}{3}\pi R^3 \vec{M} = \frac{2\pi}{\mu_0} R^3 \vec{B}$$

Thus $$\langle P_{rad} \rangle = \frac{1}{4\pi\epsilon_0} \frac{8\pi^2}{3\mu_0^2} \frac{k^4}{c} R^6 \sin^2\psi |\vec{B}|^2$$

$$= \frac{2\pi}{\mu_0} \frac{c k^4}{3} R^6 \sin^2\psi \, |\vec{B}|^2$$

Numbers: $\mu_0 = 4\pi \times 10^{-7}$ N/A, $\psi = 10^\circ = 0.17$ rad

$k = \frac{\omega}{c}$, $\omega = \frac{2\pi}{T} = 2\pi \times 10^3 \, s^{-1}$

$|\vec{B}| = 10^8$ Tesla, $R = 10^4$ m

$\Rightarrow \langle P_{rad} \rangle \cong 3 \times 10^{34}$ Watts

Note: 1 year $\approx 3 \times 10^{7}$ seconds

$$\Rightarrow \boxed{\langle P_{rad} \rangle \cong 10^{27} \text{ Joules/year}}$$

The radiated energy comes from the rotational energy of the pulsar

$$\frac{d}{dt} E_{rotation} \cong - \langle P_{rad} \rangle$$

$$E_{rotation} = \frac{2}{5} M_{sun} R^2 \omega \Rightarrow \frac{dE_{rot}}{dt} = \frac{4}{5} M_{sun} R^2 \omega \frac{d\omega}{dt}$$

$$\Rightarrow \boxed{\frac{d\omega}{dt} = - \frac{\langle P_{rad} \rangle}{\frac{4}{5} M_{sun} R^2 \omega}}$$

This is a complicated ODE because $\langle P_{rad} \rangle$ depends on ω.

We can estimate the initial rate of decay

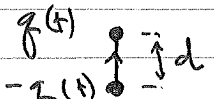
$$\Gamma = \left| \frac{\left.\frac{d\omega}{dt}\right|_0}{\left.\omega\right|_0} \right| = \frac{\langle P_{rad} \rangle}{\frac{4}{5} M_{sun} R^2 (\omega(0))^2} \approx 4.6 \times 10^{-5} / \text{year}$$

The lifetime $\boxed{\tau_{life} \approx 2 \times 10^{4} \text{ years}}$

$\tau_{life} \approx \frac{1}{\Gamma}$

Problem 3: Radiation by Antennas

(a) The idealized dipole radiator studied in Griffiths and lecture



$$I(t) = \frac{dq}{dt} = -\omega q_0 \sin\omega t$$

$$q(t) = q_0 \cos\omega t \qquad = \mathrm{Re}\big(\underbrace{-i\omega q_0}_{\tilde{I}_0} e^{-i\omega t}\big)$$

$$p(t) = q_0 d \cos\omega t$$

The dipole will radiate and carry away energy. This energy must come from a power source that drives the current.

The power lost to radiation appears to the current drive as an <u>effective resistance</u>. In a resistor, energy is lost to heat. Here it is lost to waves, but the drive can't see the difference.

$$\Rightarrow \quad \underbrace{\frac{1}{2}|\tilde{I}_0|^2 R_{rad}}_{\text{Power lost in an effective resistor}} = \langle P_{rad} \rangle$$

$$\frac{1}{2}(\omega q_0)^2 R_{rad} = \frac{1}{4\pi\epsilon_0}\frac{ck^4}{3} p_0^2 = \frac{1}{4\pi\epsilon_0}\frac{\cancel{c}\,\omega^4}{3c^3} q_0^2 d^2$$

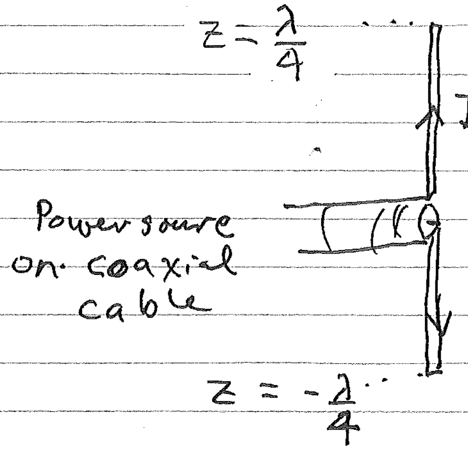
$$\Rightarrow R_{rad} = \frac{1}{6\pi\epsilon_0}\frac{\omega^2}{c^3} d^2 = \frac{1}{6\pi c\epsilon_0}\left(\frac{2\pi d}{\lambda}\right)^2$$

$\frac{\omega}{c} = 2\pi/\lambda$

$$\Rightarrow \boxed{R_{rad} = \frac{2}{3}\pi Z_0 \left(\frac{d}{\lambda}\right)^2 \approx 80\pi^2 \left(\frac{d}{\lambda}\right)^2 \text{ ohms}}$$

Z_0: impedance of free space $\approx 120\pi$ ohms

(b) In a real antenna, the current must go to zero at the ends. We consider here a "center fed" antenna



Half-wave antenna

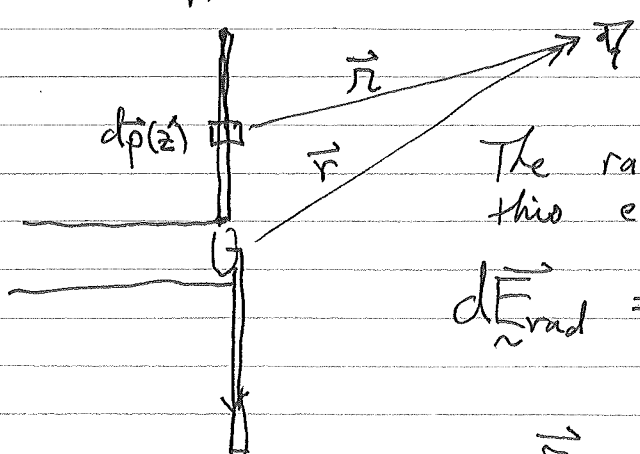
$$\Rightarrow I(z) = I_0 \cos\left(\frac{2\pi z}{\lambda}\right)$$

Because the length of the antenna $l = \frac{\lambda}{4}$ is on the order of λ, we cannot treat this as a single dipole radiator. There are two ways to deal with this:

(i) Break up the antenna into differential elements, each a dipole radiator.

(ii) Go back to first principles, find the potentials, and find the fields.

Consider approach (i) first



The radiation emitted from this element of the antenna

$$d\underset{\sim}{\vec{E}}_{rad} = \frac{1}{4\pi\epsilon_0} k^2 \, d\underset{\sim\perp}{\vec{p}}(z') \, \frac{e^{ik\mathscr{r}}}{\mathscr{r}}$$

$$\vec{\mathscr{r}} = \vec{r} - z'\hat{z}$$

- In the far field, $\mathscr{r} = |\vec{\mathscr{r}}| \approx r - \hat{r}\cdot\vec{r}\,' = r - (\hat{r}\cdot\hat{z})\,z'$

$$= r - \cos\theta\, z'$$

$$\frac{e^{ik\mathscr{r}}}{\mathscr{r}} \approx \frac{e^{ikr}\, e^{-ikz'\cos\theta'}}{r - \underbrace{\cancel{\cos\theta z'}}_{\text{negligible}}}$$

- The differential dipole $d\vec{p}$

dz' : $+q(z,t)$, $-q(z,t)$

$$\frac{dq(z,t)}{dt} = I(z,t) \Rightarrow -i\omega\tilde{q}(z) = \tilde{I}(z)$$

$$\Rightarrow \tilde{q}(z) = i\,\frac{\tilde{I}(z)}{\omega}$$

$$\Rightarrow d\tilde{\vec{p}}(z') = \tilde{q}(z')\,dz'\,\hat{z} = i\,\frac{dz'}{\omega}\,\tilde{I}(z')\,\hat{z}$$

$$d\tilde{\vec{p}}_{\perp}(z') = i\,\frac{dz'}{\omega}\, I(z')\,\sin\theta\,\hat{r}$$

$$\Rightarrow d\tilde{\vec{E}}_{rad} = \frac{1}{4\pi\epsilon_0}\,\frac{i\omega}{c^2}\,\frac{e^{ikr}}{r}\, I(z')\, e^{-ikz'\cos\theta}\,\sin\theta\,\hat{r}$$

The total field is then the integral over the antenna:

$$\boxed{\tilde{\vec{E}}_{rad} = \underbrace{\frac{1}{4\pi\epsilon_0}\,\frac{ik}{c}\,\frac{e^{ikr}}{r}\,\sin\theta\int_{-\lambda/4}^{\lambda/4} I(z')\, e^{-ikz'\cos\theta}\, dz'}_{|\tilde{\vec{E}}_{rad}|}\;\hat{r}}$$

(ii) Method – Find potentials, then $\vec{E}$

We know: $\vec{E}(\vec{r},t) = -\vec{\nabla}V - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$

For monochromatic fields: $\underset{\sim}{\vec{E}}(\vec{r}) = -\vec{\nabla}\underset{\sim}{V} + i\omega \underset{\sim}{\vec{A}}(\vec{r})$

In the far field the effect of the scalar potential is to remove the longitudinal component

$$\Rightarrow \boxed{\underset{\sim}{\vec{E}}_{rad} = i\omega \underset{\sim}{\vec{A}}^{rad}_{\perp}(\vec{r})}$$

Thus, all we have to find is the vector potential

Recall
$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}', t_{ret})}{\mathscr{r}} \qquad \text{where } \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$
$$t_{ret} = t - \frac{c}{\mathscr{r}}$$

Here the current is along a line

$$\Rightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\lambda/4}^{\lambda/4} dz' \frac{I(z', t_{ret})}{\mathscr{r}}\,\hat{z}$$

Complexify:

$$\underset{\sim}{\vec{A}}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\frac{\lambda}{4}}^{\lambda/4} dz'\, I(z')\, \frac{e^{-i\omega(t-\frac{\mathscr{r}}{c})}}{\mathscr{r}}\,\hat{z}$$

$$\Rightarrow \underset{\sim}{\vec{A}}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{-\lambda/4}^{\lambda/4} I(z')\, \frac{e^{ik\mathscr{r}}}{\mathscr{r}}\,\hat{z}$$

In radiation zone: $\mathscr{r} \simeq r - z'\cos\theta$

$$\underset{\sim}{\vec{A}}^{rad}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{-\frac{\lambda}{4}}^{\lambda/4} I(z')\, \underline{e^{-ikz\cos\theta}}\, \frac{e^{ikr}}{r}\,\hat{z}$$

Thus: $\underset{\sim}{\vec{E}}_{rad} = i\omega \underset{\sim}{\vec{A}}_{\perp}^{rad}$, $\hat{z}_{\perp} = \sin\theta\,\hat{r}$

$$= \frac{\mu_0}{4\pi}\, i\omega \int_{-\lambda/4}^{\lambda/4} I(z')\, e^{-ikz'\cos\theta}\, \frac{e^{ikr}}{r}\, \sin\theta\,\hat{r}$$

$$\Rightarrow \boxed{\underset{\sim}{E}_{rad} = \frac{1}{4\pi\epsilon_0}\, \frac{ik}{c} \sin\theta\, \frac{e^{ikr}}{r} \int_{-\lambda/4}^{\lambda/4} I(z'')\, e^{-ikz'\cos\theta}\, dz''}$$

Having used $\mu_0\epsilon_0 = \frac{1}{c^2}$ $\quad k = \frac{\omega}{c}$

(c) We must do the integral

$$d = \int_{-\lambda/4}^{\lambda/4} I(z')\, e^{-ikz'\cos\theta}\, dz' = \int_{-\lambda/4}^{\lambda/4} I_0 \underbrace{\cos\left(\frac{2\pi z'}{\lambda}\right)}_{\Downarrow \; \frac{e^{ikz'} - e^{-ikz'}}{2}}\, e^{-ikz'\cos\theta}\, dz'$$

After a little algebra

$$d = \frac{2}{k}\, \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

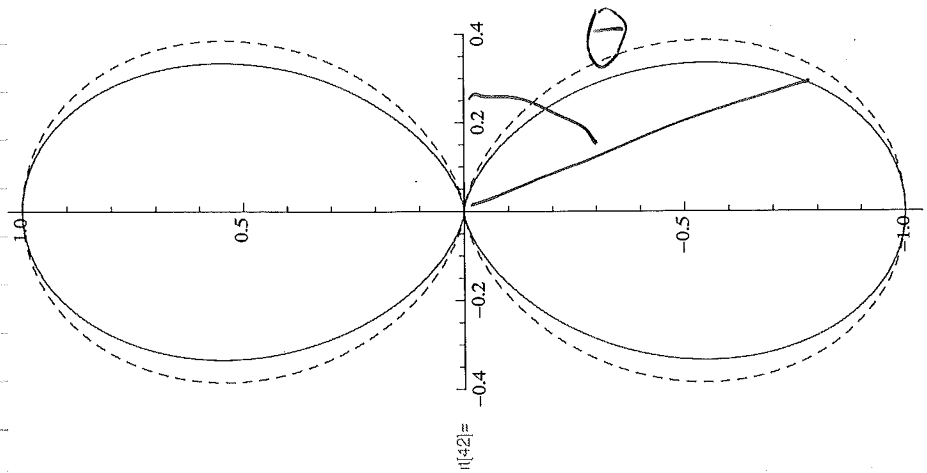
$$\Rightarrow \boxed{\underset{\sim}{E}_{rad} = \frac{1}{4\pi\epsilon_0}\, \frac{2iI_0}{c}\, \frac{e^{ikr}}{r}\, \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}$$

This is equivalent to the form given in the assignment with $l = \frac{\lambda}{2}$

(d) The time-averaged Power squared / solid angle

$$\frac{dP_{rad}}{d\Omega} \propto |\tilde{E}_{rad}|^2$$

$$\propto \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



Above is a polar plot made with Mathematica. The dashed line is the standard dipole pattern $\sin^2\theta$. We see that the half-wave antenna concentrates the power into a tigher beam @ $\theta = 90°$.

(e) We now want to integrate the radiated power over all solid angles to get the total

$$P_{rad} = \int d\Omega \frac{dP_{rad}}{d\Omega} = \int 2\pi\, d(\cos\theta) \frac{dP_{rad}}{d\Omega}(\cos\theta)$$

$$= 2\pi \int_{-1}^{1} d\mu \frac{dP_{rad}}{d\Omega}(\mu) \qquad \text{where } \mu = \cos\theta$$

$$\frac{dP_{rad}}{d\Omega} = \langle \vec{S} \rangle \cdot \hat{r}\, r^2 = \frac{c\epsilon_0}{2} \left| \vec{\underset{\sim}{E}}_{rad} \right|^2 r^2$$

$$= \frac{I_0^2}{8\pi^2 c\, \epsilon_0} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$\Rightarrow P_{rad} = \int \frac{dP_{rad}}{d\Omega} = \frac{1}{8\pi} Z_0 I_0^2 \int_{-1}^{1} 2\pi\, d\mu \frac{\cos^2\left(\frac{\pi}{2}\mu\right)}{1-\mu^2}$$

$$= \frac{1}{4\pi} Z_0 I_0^2 \underbrace{\int_{-1}^{1} d\mu \frac{\cos^2\left(\frac{\pi\mu}{2}\right)}{1-\mu^2}}_{\text{Numerically } \cong 1.22} = \frac{1}{2} I^2 R_{rad}$$

$$\Rightarrow \boxed{R_{rad} = \frac{0.61}{\pi} Z_0 \approx 73 \text{ ohms}}$$

To impedance match, this must be equal to the impedance of the coax.

In comparison, in part (a) an ideal dipole antenna has a radiation resistance

$$R_{rad}^{(dipole)} = \frac{2}{3}\pi Z_0 \left(\frac{d}{\lambda}\right)^2 = \frac{\pi}{6} Z_0 \quad \left(\text{for } d = \frac{\lambda}{2}\right)$$

$$= 197 \text{ ohms}$$ (more than a factor of 2 different)