Physics 491: Lecture #2

Probability: Quantum vs. Classical

More ~~off~~ on the "Born Interpretation" of wave function

Schrödinger wave function $\Psi(\vec{x},t)$

$P(\vec{x},t) \equiv |\Psi(\vec{x},t)|^2 d^3x$ = Probability to detect the particle (at time t) in a volume $d^3x = dx\,dy\,dz$ around the position $\vec{x}=(x,y,z)$.

Where in the world did Max Born come up with this? To gain some insight into this, consider the classical wave analog.

Consider and electromagnetic wave that is "quasimonochromatic"

$$E(\vec{x},t) = \underbrace{E_0(t)}_{\text{Slowly varying amplitude}} \cos(\omega t + \phi(\vec{x}))$$

$$= \tfrac{1}{2} E_0(\vec{x},t)\, e^{i\phi(\vec{x})}\, e^{-i\omega t} + \underbrace{c.c.}_{\text{complex conjugate}}$$

$$\equiv \mathrm{Re}\left(\mathcal{E}(\vec{x},t)\, e^{-i\omega t} \right)$$

where $\mathcal{E}(\vec{x},t) = E_0(\vec{x},t)\, e^{i\phi(\vec{x})}$ complex amplitude

Intensity = Flux energy density

$$I(\vec{x},t) \equiv \frac{c}{8\pi}\overline{(E^2+B^2)} = \frac{c}{4\pi}\overline{(E^2(\vec{x},t))}$$

where $\overline{(\ )} = \int_0^T dt\ (\ )$ = time average over short period $T = \frac{2\pi}{\omega}$

Aside:

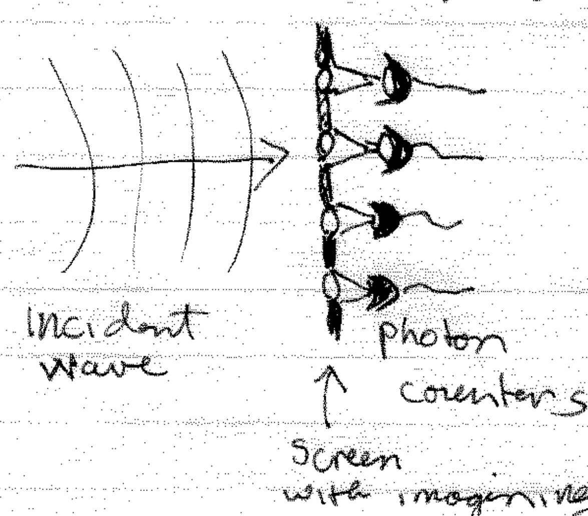
$$\overline{E^2(\vec{x},t)} = \int_0^T dt\ E_0^2(\vec{x},t)\ \cos^2(\omega t + \phi(\vec{x}))$$

$$\approx E_0^2(\vec{x},t) \underbrace{\int_0^T dt\ \cos^2(\omega t + \phi(\vec{x}))}_{1/2}$$

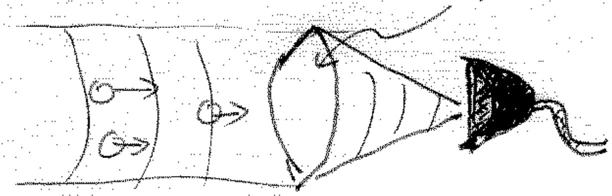
$$\Rightarrow \overline{E^2(\vec{x},t)} = \frac{1}{2} E_0^2(\vec{x},t) = \frac{1}{2}\|\mathcal{E}(\vec{x},t)\|^2$$

$\therefore$ $\boxed{\text{Square of magnitude of complex amplitude } \mathcal{E}^*(\vec{x},t)\,\mathcal{E}(\vec{x},t) \propto \text{Wave intensity}}$

Now let us think of a pulse of light as beam of photons (roughly)



Zoom into single counter



Assuming perfect detection efficiency

$$\mathcal{R} = \frac{P(t)}{\hbar\omega}$$

where $P(t)$ is the power $\left(\frac{\text{Energy}}{\text{time}}\right)$ of the beam

$$P(t) = \int_A da\, I(\vec{x},t) \quad \left(\begin{array}{c}\text{Flux through}\\ \text{aperature}\end{array}\right)$$

Average number detected in time window τ

$$N = \int_\tau dt \int_A da\, \frac{I(\vec{x},t)}{\hbar\omega} = \frac{1}{8\pi}\int c\,dt \int da\, \frac{|\mathcal{E}(\vec{x},t)|^2}{\hbar\omega}$$

$$= \int_V d^3x\, \frac{|\mathcal{E}(\vec{x},t)|^2}{8\pi\hbar\omega} = \int_V d^3x\, \boxed{\frac{\mathcal{U}(\vec{x},t)}{\hbar\omega}} = N_V$$

Photon density in volume $(c\tau A) = V$

The total number of the photons in the pulse

$$N_{total} = \int_{\substack{\text{all}\\ \text{space}}} d^3x\, \frac{|\mathcal{E}(\vec{x},t)|^2}{8\pi\hbar\omega}$$

Let us define

$$\tilde{\Psi}(\vec{x},t) \equiv \frac{\mathcal{E}(\vec{x},t)}{\sqrt{8\pi\hbar\omega}} \quad \left(\begin{array}{l}\text{The "unnormalized} \\ \text{wave function"}\end{array}\right)$$

$\Rightarrow$ Number in volume V | Total

$$N_V = \int_V d^3x \, |\tilde{\Psi}(\vec{x},t)|^2 \qquad N_{tot} = \int_{\text{all space}} d^3x \, |\tilde{\Psi}|$$

Fraction in volume $\mathcal{F}_V = \dfrac{N_V}{N_{tot}}$

Define normalized wave function

$$\Psi(\vec{x},t) = \frac{\tilde{\Psi}(\vec{x},t)}{\sqrt{N_{total}}} = \frac{\tilde{\Psi}(\vec{x},t)}{\sqrt{\int d^3x \, |\tilde{\Psi}(\vec{x},t)|^2}}$$

$\Rightarrow$ Fraction detected in volume d^3x

$$\boxed{\mathcal{F}_{d^3x} = |\Psi(\vec{x},t)|^2 d^3x} \Leftarrow \text{Born probability!}$$

Fraction of events $\Longrightarrow$ probabilities

Roughly: Given N_{tot} random events the number which land in V

$$N_V = \mathcal{F}_V N_{total} \longrightarrow \underset{\substack{\uparrow \\ \text{Probability}}}{P_V} N_{total}$$

Understanding probabilities

The connection between probabilities and frequency of events of "identical prepared random variables" is a subtle mathematical issue in statistics (we'll look at this next lecture). However, the notion of probability is a separate idea, distinct from the frequency of events.

For example, when the weather person says there is a 30% chance of rain tomorrow, (s)he is talking only about one event, the weather tomorrow. If someone flips a coin and you assign a 50-50 chance of heads, you are assuming the coin is fairly weighted, and that the flipper can't bias his throw towards heads (I know someone who can do this).

Two points are clear:

(1) Probabilities are assigned to individual events

(2) The assignment of probability depends on prior knowledge, i.e. what you know or assume.

This is the $\boxed{\text{Bayesian view}}$ *

The rules of probability are thus logical deductions in the face of <u>incomplete information</u>. For example, as the weather person refines his/her model we expect better predictive power (probabilities closer to 0 or 1

false true

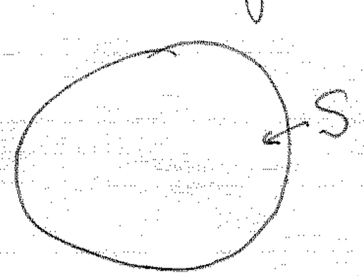
not maybe).

If we know exactly the initial conditions of a toss, the air pressure, etc., we should be able to <u>predict exactly</u> the result, heads or tails. However, since we don't have all the information, we must resort to probabilities.

A real crusader on this point of view was Ed Jaynes, author of "Probability theory: the Logic of Science", just reprinted.

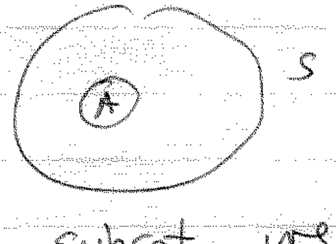
To better understand this, let us review the basic <u>axiomatic structure</u>

Consider a set S (say finite # elements) each representing a possible outcome of some event (e.g. die with six sides ⇒ six outcomes)



Sample Space

An outcome A is a subset (or member) of S

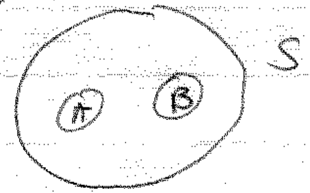


To each subset we assign a probability $P(A)$ with the follow rules.

(i) $P(A) \geq 0$ (ii) $P(S) = 1$

(iii) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

(A or B)



A and B are "mutually exclusive"

Note: If $\bigcup_i (A_i) = S \Rightarrow \sum_i P(A_i) = 1$ "normalization"

Note: If S has N elements and all are equally likely $P(A_i) = \frac{1}{N}$. If $B = \bigcup_{i=1}^{M} A_i$

the $P(B) = \frac{M}{N}$ (Bernoulli rule)

Assigning probabilities to outcomes is often an issue of counting the number M of possibilities associated with subset B given a total number of possibilities N.

Joint ~~prob~~ probabilities and conditionals

~~Let~~ $P(A \text{ and } B) = P(A \cap B) \equiv P(A, B)$

joint prob.

Define conditional probability

$$P(A|B) \equiv P(A \text{ given } B \text{ is true})$$

$$P(A, B) \equiv P(A|B)\,P(B)$$

$$= P(B|A)\,P(A)$$

Statistically independent events (uncorrelated)

$P(A|B)$ does not depend on B

$$\Rightarrow \quad P(A, B) = P(A)\,P(B)$$

Note: If $B = \bigcup_{i=1}^{M} B_i \leftarrow$ union of M exclusive alternatives

$$P(A, B) = P\left(A \cap \left(\bigcup_i B_i\right)\right) = P\left(\bigcup_i (A \cap B_i)\right)$$

$$= \sum_i P(A \cap B_i) = \sum_i P(A, B_i)$$

$$\boxed{P(A, B) = \sum_i P(A|B_i)\,P(B_i)}$$

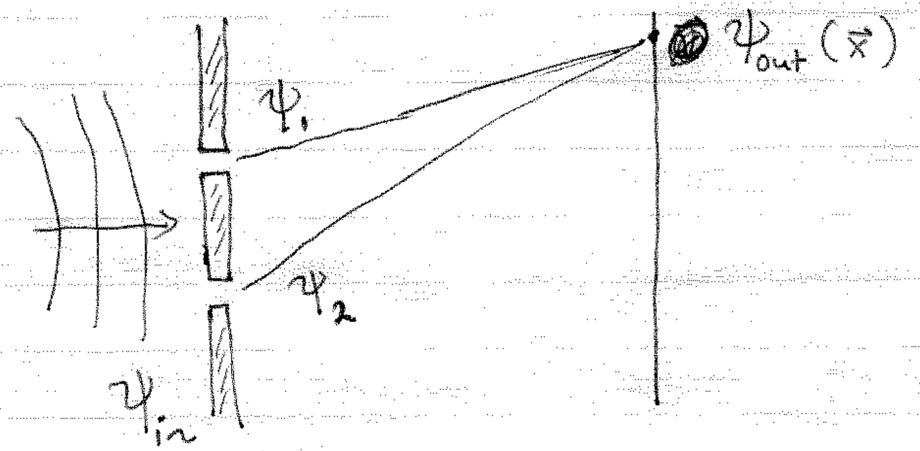
↑ "Common sense"

i.e.

intuitive

Counter-intuitive Quantum Probability

Consider again the double slit experiment



There seem to be two mutually exclusive alternatives. Either the photon goes through slit 1 or 2. By intuitive logic, the probability to be detected at point $\vec{x}$ (with d^3x)

$$P(\vec{x}) = \underbrace{P_1(\vec{x})}_{\substack{\text{prob. to go} \\ \text{through 1 and land at } \vec{x} \to \vec{x}+d^3x}} + P_2(\vec{x})$$

However, by Born rule:

$$P(\vec{x}) = |\psi_{out}(\vec{x})|^2 d^3\vec{x}$$

And by the principle of superposition

$$\psi_{out}(\vec{x}) = \psi_1(\vec{x}) + \psi_2(\vec{x})$$

(Next Page)

$$\Rightarrow |\psi_{out}|^2 d^3\vec{x} = \left(\psi_{out}^* \psi_{out}\right) d^3\vec{x}$$

$$= (\psi_1^* + \psi_2^*)(\psi_1 + \psi_2)\, d^3x$$

$$= |\psi_1|^2 d^3x + |\psi_2|^2 d^3x$$

$$+ \left(\psi_1^* \psi_2 + \psi_2^* \psi_1\right) d^3x$$

$$\Rightarrow \boxed{P(\vec{x}) = \underbrace{P_1(\vec{x}) + P_2(\vec{x})}_{\text{intuitive part}} + \underbrace{2\,\mathrm{Re}(\psi_1^* \psi_2)\, d^3x}_{\text{Interference}}}$$

The wave function is sometimes known as the "probability amplitude". It is calculational device for finding probabilities which are the squares of the amplitudes. Through the superposition of amplitudes, interference occurs between two paths that lead to the same final outcome. This interference can be constructive or destructive, increasing or decreasing the probability of a given event.

Complementarity and Wave-particle Duality

Bohr was the intellectual leader who formulated much of the way we think ~~of~~ about the quantum world. The principles became known as the "Copenhagen interpretation", as much of the work was done there in Denmark. Though much of this has held up today, Bohr's perspective is sometimes hard to put into words. One of the cornerstones is the so-called "complementarity principle". Loosely (the best I can do), quanta have complementary aspects: particles and waves. An experiment can show one or the other feature, but not both. To determine which one must take the <u>whole apparatus</u> into consideration. It is only through measurement that we learn about a system. Thus, if we try to <u>measure</u> which slit a photon goes through we <u>remove the interference</u>. The act of measuring has changed ~~our~~ quantum system and removed the interference. While this might seem like "magic" it has been borne out in experiments. We'll have much more to say about this as the semester progresses.