Physics 491

Lecture 4: Review of Wave Theory

We have discussed ~~the~~ "Born interpretation" whereby $|\Psi(\vec{x},t)|^2$ is the probability density for finding a particle (e.g. an electron) near the position $\vec{x}$. What about other observables ~~th~~ like momentum, energy, or angular momentum? How do we extract that information from Ψ? Given the wave function $\Psi(\vec{x},t_0)$ how do we determine its value for times $t > t_0$? The answer to these questions fall under the general heading of "wave mechanics" as named by Schrödinger in his formulation of the quantum theory. ~~This~~ This proceeded simultaneously with Heisenberg's "matrix mechanics" formulation, the two shown soon after to be equivalent – just different representations of the same abstract theory. We will spend the majority of this semester dealing with ~~matrix~~ wave mechanics, returning to matrix mechanics next semester.

Waves review

Consider waves in one dimension (e.g. waves on a string), with amplitude $F(x,t)$

The wave equation is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)F(x,t) = 0$$

An important class of ~~problems~~ solutions are the "plane waves"

$$F(x,t) = A\cos(kx - \omega t + \phi)$$

Check that this is a solution:

$$\frac{\partial^2 F}{\partial t^2} = -\omega^2 F \qquad \frac{\partial^2 F}{\partial x^2} = -k^2 F$$

$$\Rightarrow \left(-k^2 + \frac{\omega^2}{v^2}\right) F(x,t) = 0$$

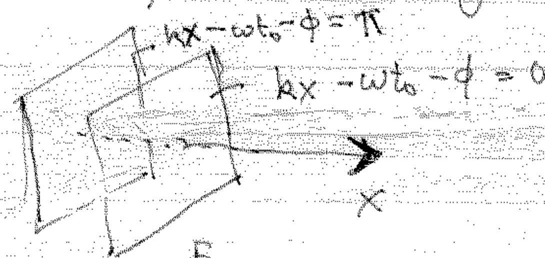
The solution requires $k^2 = \frac{\omega^2}{v^2}$ or $k = \frac{\omega}{v}$

The relation $\omega(k)$ is known as the "dispersion relation"

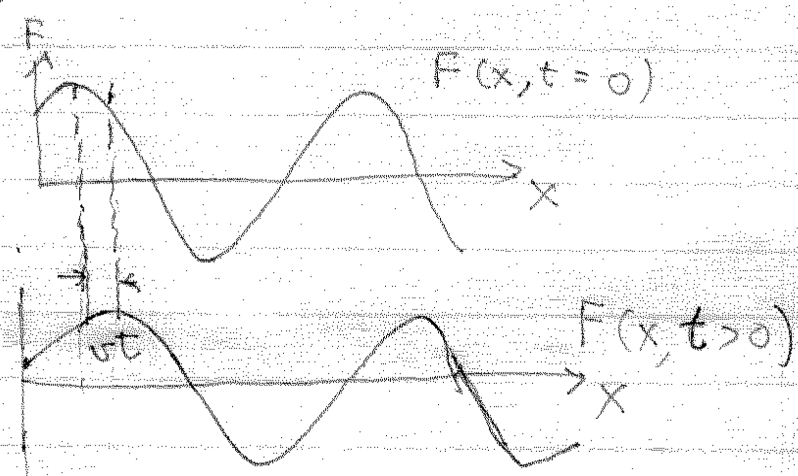
This is known as a plane wave because at any time t_0, the surface of constant phase

i.e. $kx - \omega t_0 + \phi = \text{constant}$

Are planes, here the y-z plane $\perp$ to x



At $t = 0$



$t > 0$

The planes of constant phase propagate in the x-direction at velocity $v = \frac{\omega}{k}$

$$F(x,t) = A\cos\big(k\underbrace{(x - \tfrac{\omega}{k}t)}_{= x - vt} + \phi\big)$$

$v = \frac{\omega}{k}$ is known as the <u>phase velocity</u>

<u>Complex notation:</u> In dealing with sinusoidal functions we often use complex numbers

$$F(x,t) = \mathrm{Re}\big(\underbrace{Ae^{i\phi}}_{\tilde{F}_0 \leftarrow \text{complex amplitude}}\, e^{i(kx-\omega t)}\big)$$

Complex plane wave $\quad F(x,t) = \tilde{F}_0\, e^{i(kx-\omega t)}$

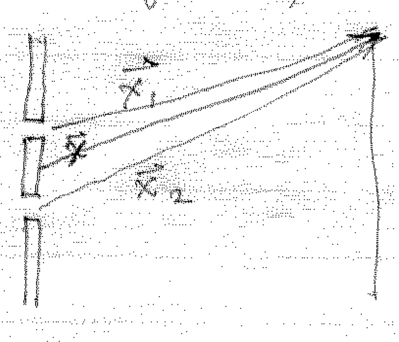
<u>Interference:</u> $\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) \equiv \Box$ is

a <u>linear operator</u>: $\quad \Box\,(c_1 F_1(x,t) + c_2 F_2(x,t))$

$$= c_1(\Box F_1) + c_2(\Box F_2)$$

$\Rightarrow$ If $F_1(x,t)$ and $F_2(x,t)$ are solutions to the wave equation, so is a <u>superposition</u>

e.g. double slit



Far away from the slits, the waves emanating from them look like plane waves.

Very far from the slits $\vec{x}_1$ and $\vec{x}_2$ are approximately parallel (let's call this the x-direction).
The total wave amplitude incident at $\vec{x}$ is

$$F(x,t) = F_1(x_1,t) + F_2(x_2,t)$$

where $F_1(x_1,t) = F_0 e^{i(kx_1-\omega t)}$ $\quad F_2(x_2,t) = F_0 e^{i(kx_2-\omega t)}$

The total intensity

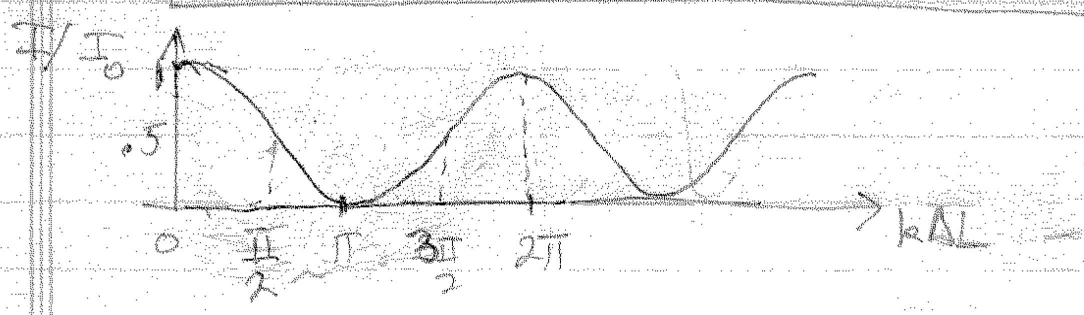
$$I \sim |F(x,t)|^2 = F^*(x,t)\, F(x,t)$$

$$= |F_1(x_1,t)|^2 + |F_2(x_2,t)|^2 + 2\,\mathrm{Re}\left(F_1^*(x_1,t)\, F_2(x_2,t)\right)$$

$$= \underbrace{|F_0|^2}_{\equiv I_0} + |F_0|^2 + 2\,\mathrm{Re}\left(|F_0|^2 e^{ik(x_2-x_1)}\right)$$

$$= 2I_0\left(1 + \cos k(x_2-x_1)\right)$$

$$\Rightarrow \boxed{I = 4I_0 \cos^2\left(\frac{k\Delta L}{2}\right) \quad \text{where} \quad \Delta L = x_2 - x_1}$$



Note: When $k\Delta L = m\pi$ (m odd integer)

$$\Delta L = m\frac{\pi}{k} = m\left(\frac{\lambda}{2}\right) \Rightarrow \text{Destructive interference}$$

$$k\Delta L = m(2\pi),\ \Delta L = m\lambda \Rightarrow \text{Constructive interference}$$

Wave packets

Consider now a superposition of waves of different frequencies and wavelengths

$$F(x,t) = F_0\, e^{i(k_1 x - \omega_1 t)} + F_0\, e^{i(k_2 x - \omega_1 t)}$$

$$= F_0 \left(e^{i\Phi_1} + e^{i\Phi_2} \right) \quad \text{where } \Phi_1 = k_1 x - \omega_1 t, \quad \Phi_2 = k_2 x - \omega_2 t$$

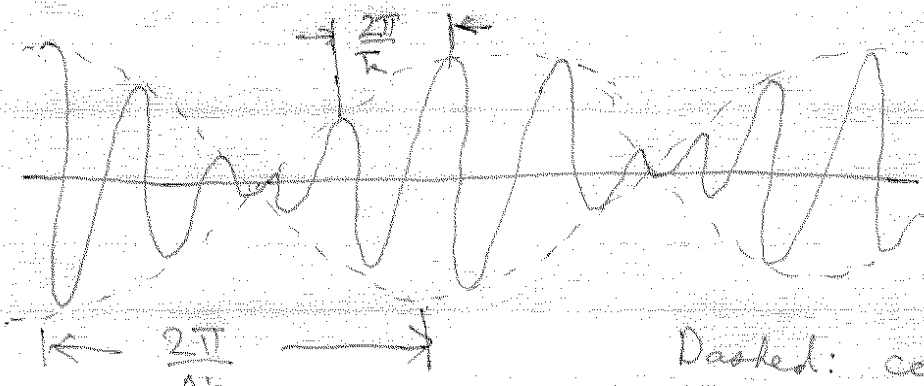
$$= F_0\, e^{i\frac{(\phi_1 + \phi_2)}{2}} \Big(\underbrace{e^{i(\phi_1 - \phi_2)/2} + e^{-i(\phi_1 - \phi_2)/2}}_{2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)} \Big)$$

$$\therefore\ F(x,t) = F_0\, e^{i(\bar{k}x - \bar{\omega}t)} \cos(\Delta k x - \Delta\omega t)$$

$$\left(\text{where } \bar{k} = \frac{k_1 + k_2}{2}, \quad \Delta k = k_1 - k_2, \text{ etc.} \right)$$

Or taking the real part

$$F(x,t) = F_0 \cos(\Delta k x - \Delta\omega t)\, \cos(\bar{k}x - \bar{\omega}t)$$



Dashed: $\cos(\Delta k x - \Delta\omega t)$
Solid: $F(x,t)$

<u>Note:</u> $\Delta k < \bar{k} \;\Rightarrow\; \frac{2\pi}{\Delta k} > \frac{2\pi}{\bar{k}}$

The term $\cos(\Delta k x - \Delta\omega t)$ creates an "envelope" which modulates the amplitude of the wave

Note: The envelope propagates at the velocity $\frac{\Delta\omega}{\Delta k}$ whereas the phase propagates at $\frac{\bar{\omega}}{\bar{k}}$

$\frac{\Delta\omega}{\Delta k}$ is known as the <u>Group velocity</u>

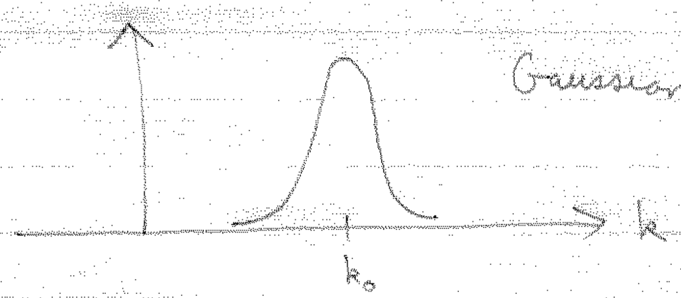
Now suppose we add up many waves of different wavelength. Consider this at $t=0$ for a <u>continuum</u> of waves

$$F(x,0) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{F}(k)\, e^{ikx}$$

← (convention)

Through an appropriate choice of superposition amplitudes we can create a localized <u>wave packet</u>

Example: $\tilde{F}(k) = F_0\, e^{-(k-k_0)^2/\alpha^2}$



Gaussian centered at k_0

$$\Rightarrow F(x,0) = F_0 \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{-(k-k_0)^2/\alpha^2}\, e^{ikx}$$

$$= F_0\, e^{ik_0 x} \int_{-\infty}^{\infty} \frac{dk'}{\sqrt{2\pi}}\, e^{-k'^2/\alpha^2}\, e^{ik'x}$$

Trick for dealing with Gaussian integrals: "Complete the square"

Aside: $-\frac{k'^2}{\alpha^2} + ik'x = -\frac{1}{\alpha^2}\left(k'^2 - i\alpha^2 x k'\right)$ (Next Page)

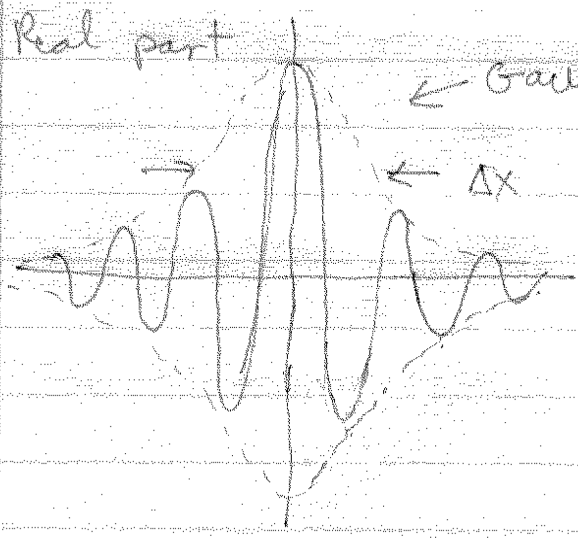
Now, $k'^2 - Ak' = \left(k' - \frac{A}{2}\right)^2 - \frac{A^2}{4}$

$$\Rightarrow -\frac{k'^2}{\alpha^2} - ik'x = -\frac{1}{\alpha^2}\Big(k' - i\underbrace{\frac{\alpha^2 x}{2}}_{\equiv A/2}\Big)^2 - \frac{\alpha^2}{4}x^2$$

$$\therefore\ F(x,0) = F_0\, e^{-\frac{\alpha^2 x^2}{4}} \underbrace{\left[\int_{-\infty}^{\infty} e^{-\frac{(k'-\frac{A}{2})^2}{\alpha^2}} \frac{dk'}{\sqrt{2\pi}}\right]}_{=\sqrt{2\pi\alpha^2}\ \text{even when } A \text{ complex}} e^{ik_0 x}$$

$$\Rightarrow F(x,0) = \underbrace{F_0'\, e^{-\frac{\alpha^2 x^2}{4}}}_{\text{Gaussian envelope}}\ e^{ik_0 x}$$

phase



← Gaussian wave packet

Note $\Delta x \sim \frac{1}{\alpha}$

This is our first encounter with the "uncertainty principle" — More next lecture

The essence: To ~~get~~ obtain a very localized wave packet we much take a super-position of a broad spectrum of waves, $\Delta k \sim \alpha$ large

<u>Propagation of wave packets</u>

Given $F(x,0) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{F}(k)\, e^{ikx}$

Each component evolves as $\tilde{F}(k)\, e^{i(kx - \omega(k)t)}$

Take the superposition

$$\Rightarrow F(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{F}(k)\, e^{i(kx - \omega(k)t)}$$

This is our first example of the solution to the <u>initial value problem</u>. Because plane waves have a simple time dependence we can easily solve for $F(x,0)$ by expressing as a superposition of the <u>normal modes</u>.

Suppose we have a wave packet where $\tilde{F}(k)$ is peak around k_0 with a narrow width (as in our Gaussian wave packet example). The we can expand $\omega(k)$ in a Taylor series

$$\omega(k) = \omega(k_0) + \left.\frac{\partial\omega}{\partial k}\right|_{k_0} (k - k_0) + \cdots$$

$$\Rightarrow kx - \omega t = (k_0 x - \omega_0 t) + (k - k_0)\Big(x - \underbrace{\left.\frac{\partial\omega}{\partial k}\right|_{k_0}}_{= v_g} t\Big) + \cdots$$

$$\Rightarrow F(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} \tilde{F}(q + k_0)\, e^{iq(x - v_g t) + \cdots}$$

If we ignore the "..." terms the packet travels at the <u>group velocity</u> $v_g = \left.\frac{\partial\omega}{\partial k}\right|_{k_0}$. The neglected terms distort the packet through <u>dispersion</u>

Fourier Transform

Suppose we are given a wavepacket $F(x,0) \equiv F(x)$. To determine the expansion in terms of plane waves we appeal to Fourier's theorem:

Any function which is (piecewise) continuous and "absolutely integrable" $\int_{-\infty}^{\infty} dx \, |F(x)| < \infty$ can be expressed as superposition of plane waves

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{F}(k) \, e^{ikx}$$

where

$$\boxed{\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} F(x) \, e^{-ikx}}$$ Fourier transform

Consistency

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \underbrace{\left[\int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}} F(x') \, e^{-ikx'} \right]}_{\tilde{F}(k)} e^{ikx}$$

Reverse order of integration (OK for class of functions above)

$$F(x) = \int_{-\infty}^{\infty} dx' \, F(x') \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}}_{\equiv \, \delta(x-x') \,:\, \text{Dirac delta function}}$$

$$\Rightarrow \boxed{F(x) = \int_{-\infty}^{\infty} dx' \, F(x') \, \delta(x-x')}$$

Loosely, $\delta(x-x')$ "picks off" the value $x = x'$

Note: if $F(x) = 1$

$$\Rightarrow \boxed{1 = \int_{-\infty}^{\infty} \delta(x-x')\, dx' = \int_{-\infty}^{\infty} \delta(x-x')\, dx}$$

"Normalized" area $= 1$

Explicity expression for delta-function

$$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}$$

$$= \left.\frac{e^{ik(x-x')}}{2\pi i k}\right|_{k=-\infty}^{k=+\infty} = ??$$

To evaluate the integral, introduce a "regularizing" factor $e^{-\epsilon|k|}$ $\epsilon > 0$ and take time $\epsilon \to 0_+$

$$\delta(x-x') = \lim_{\epsilon \to 0_+} \int_{-\infty}^{\infty} e^{ik(x-x') - \epsilon|k|}\, \frac{dk}{2\pi}$$

$$= \lim_{\epsilon \to 0_+} \left(\int_{-\infty}^{0} \frac{dk}{2\pi} e^{ik(x-x') + \epsilon k} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ik(x-x') - \epsilon k} \right)$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon + i(x-x')} - \frac{1}{-\epsilon + i(x-x')} \right] \frac{1}{2\pi}$$

$$\Rightarrow \delta(x-x') = \lim_{\epsilon \to 0} \underbrace{\frac{\epsilon/\pi}{\epsilon^2 + (x-x')^2}}_{\equiv\, \delta_\epsilon(x-x')}$$

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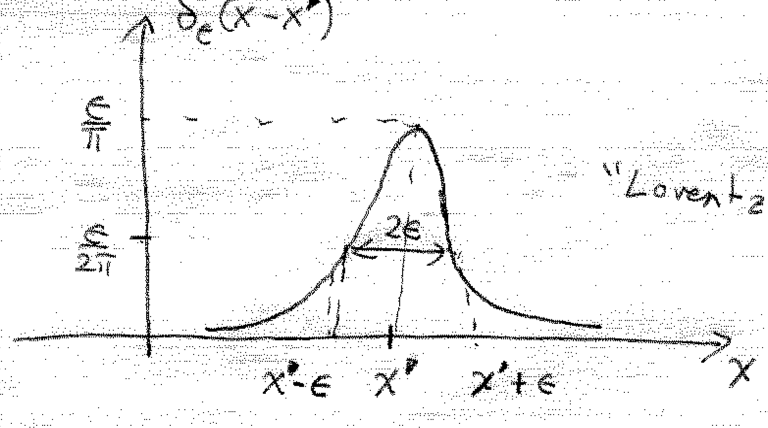
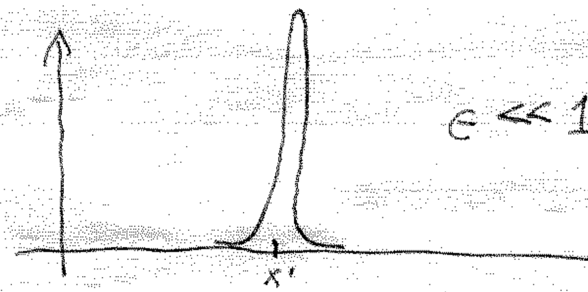


"Lorentzian"

easy to show $\int_{-\infty}^{\infty} \frac{\epsilon/\pi}{\epsilon^2 + (x-x')^2}\, dx = 1$ normalized

Now, as $\epsilon \to 0$ the curve becomes more and more peaked around $x = x'$, with unit area under curve.



$\epsilon \ll 1$

In the limit $\epsilon \to 0$ $\delta_\epsilon(x-x') \to \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases}$

Thus the delta "function" is not a well behaved function at all. Nonetheless, it is well behaved inside the integral and is an important tool for dealing with waves and Fourier integrals. Formally speaking $\delta(x-x')$ is known as a "distribution", not a function.