Physics 491: Quantum Mechanics I

Lecture 6: Eigenfunctions / Eigenvalues

Solutions to the Schrödinger Equation

The T.D.S.E. : $\frac{\hbar}{-i} \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\, \Psi(x,t)$

This is a partial diff'eq. We can use the method of separation of variables to find solutions

Let $\Psi(x,t) = u(x)\, f(t)$

$$\Rightarrow \underbrace{\frac{\hbar}{-i f(t)} \frac{df(t)}{dt}}_{\equiv\, E} = \underbrace{\frac{1}{u(x)} \left[-\frac{\hbar^2}{2m} \frac{d^2u}{dx^2} + V(x)\, u(x) \right]}_{E}$$

"separation constant"

$$\Rightarrow \frac{df_E(t)}{dt} = -\frac{i}{\hbar} E\, f_E(t) \quad \Rightarrow f_E(t) = f_0\, e^{-\frac{i}{\hbar} E t}$$

f_0 will be fixed by normalization

$$\Rightarrow \Psi_E(x,t) = u_E(x)\, e^{-\frac{i}{\hbar} E t}$$

Note: $|\Psi_E(x,t)|^2 = |u_E(x)|^2$ Independent of time

Thus these are stationary states

The functions $u_E(x)$ satisfy

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2u_E}{dx^2} + V(x)\,u_E(x) = E\,u_E(x)}$$

This is known as the time *independent* schrödinger equation (T.I.S.E.)

Often the T.I.S.E is written as

$$\boxed{\hat{H}\,u_E(x) = E\,u_E(x)}$$

where $\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$$= \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 + V(x)$$

$$\Rightarrow \boxed{\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \equiv \underline{\underline{\text{Hamiltonian}}}}$$

In classical physics the Hamiltonian is the <u>energy</u> of the system. The stationary states are thus states of definite <u>energy</u> (no uncertainty in energy). In wave mechanics they oscillate as $e^{-iEt/\hbar} = e^{-i\omega t}$, $\hbar\omega = E$. Thus are thus the <u>normal</u> <u>modes</u> of system.

Eigenfunctions

The equation of the form

$$\hat{H}\, u_E(x) = E\, u_E(x)$$

is known as an eigenfunction equation.

Eigen ≡ characteristic in German.

$$u_E(x) = \text{Eigenfunction of } \hat{H}$$

$$E = \text{Eigenvalue of } \hat{H}$$

On its eigenfunctions the Hamiltonian operator acts like multiplication by a constant.

We have already seen another eigen-equation

$$\hat{P}\, u_p(x) = p\, u_p(x)$$

$$\Rightarrow \frac{\hbar}{-i}\frac{\partial}{\partial x} u_p(x) = p\, u_p(x) \Rightarrow \frac{d u_p(x)}{dx} = \frac{-ip}{\hbar} u_p(x)$$

$$\Rightarrow \boxed{u_p(x) = u_0\, e^{-ipx/\hbar}}$$

The plane wave $u_0 e^{-ipx/\hbar}$ is an eigenfunction of the momentum operator, with eigenvalue p.

Examples of stationary states:

Free particle: $V(x) = 0 \Rightarrow \hat{H} = \frac{\hat{p}^2}{2m}$

T.I.S.E. $\Rightarrow \frac{\hat{p}^2}{2m} u_E(x) = E\, u_E(x)$

or $\frac{-\hbar^2}{2m} \frac{d^2 u_E}{dx^2} = E\, u_E(x)$

Instead of solving the diff'eq note that if $u_p(x)$ is an eigenfunction of $\hat{p}$ with eigenvalue p, then it is also an eigenfunction of $\hat{p}^2$ with eigenvalue p^2:

$$\hat{p}^2 u_p(x) = \hat{p}\left(\hat{p}\, u_p(x)\right) = \hat{p}\left(p\, u_p(x)\right)$$

$$\underset{\text{linearity}}{=} p\left(\hat{p}\, u_p(x)\right) = p^2 u_p(x)$$

$\therefore$ $u_p(x) = u_0\, e^{-ipx/\hbar}$ is an eigenfunction of $\hat{H} = \frac{\hat{p}^2}{2m}$ with eigenvalue

$$E(p) = \frac{p^2}{2m}$$

(the dispersion relation for free particle)

We say that the plane wave is a simultaneous eigenfunction of $\hat{H}$ and $\hat{p}$

Degeneracy:

Consider $u_{-p}(x) = u_0 e^{+ipx/\hbar}$

$$\hat{p}\, u_{-p}(x) = -p\, u_{-p}(x)$$

Particle with momentum $-p$

However, for a free particle $\hat{H}_{free} = \frac{\hat{p}^2}{2m}$

$$\hat{H}_{free}\, u_{-p}(x) = \frac{p^2}{2m}\, u_{-p}(x) = E(p)\, u_{-p}(x)$$

Thus $u_p(x)$ and $u_{-p}(x)$ have the same energy eigenvalue. The are said to be degenerate eigenfunctions of $\hat{H}_{free}$

Because $\hat{H}$ is a linear operator any linear combination of $u_p(x)$ and $u_{-p}(x)$ is an eigenfunction of $\hat{H}_{free}$:

Let $\psi(x) = A e^{-ipx/\hbar} + B e^{+ipx/\hbar}$

$$\hat{H}\psi = A\left(\hat{H}_{free} e^{-ipx/\hbar}\right) + B\left(\hat{H}_{free} e^{-ipx/\hbar}\right)$$

$$= E(p)\left(A e^{-ipx/\hbar} + B e^{-ipx/\hbar}\right)$$

$$= E(p)\,\psi \quad \checkmark$$

Generally, A and B will be determined by boundary