Lecture 8: Introduction to Hilbert Space

More on the probability interpretation of the expansion coefficients.

Last lecture we showed that if we considered a wave function that is a superposition of energy eigenfunctions (e.g. for ∞-well)

$$\psi(x) = \sum_n c_n u_n(x)$$

Then the expected value of the energy

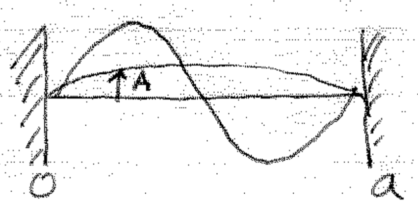
$$\langle E \rangle = \int dx\, \psi^*(x)\, \hat{H}\, \psi(x) = \sum_n P(E_n)\, \underset{\substack{\uparrow \\ \text{energy} \\ \text{eigenvalue}}}{E_n}$$

where $P(E_n) = |c_n|^2$

Thus the expansion coefficients c_n are to be interpreted as probability amplitudes and $|c_n|^2$ are the probabilities that we will find energy E_n when a measurement of energy is performed.

Why does this make sense? To better understand this, let's turn to a classical wave problem.

Consider a string of length a, pinned between rigid walls (see Marion & Thorton, Classical Mechanics)



Shown here are the first two normal modes

Let $A(x,t)$ be the displacement of the string a position x and time t.

We can decompose this function into normal modes:

$$A(x,t) = \mathrm{Re}\Big\{ \underbrace{\sum_n q_n \sin(k_n x)\, e^{-i\omega_n t}}_{\tilde{A}(x,t) \,=\, \text{Complex function}} \Big\}$$

where $k_n = n\frac{\pi}{a}$ $\quad \omega_n = v k_n \quad v = \sqrt{\frac{\tau}{\rho}}$ = phase velocity

$\rho = \frac{\text{mass}}{\text{length}} \quad \tau$ = Young's modulus

The time average energy

$$\overline{E} = \text{Kinetic} + \text{Potential}$$

$$= \int_0^a dx \left\{ \frac{\rho}{2} \left| \frac{\partial \tilde{A}}{\partial t} \right|^2 + \frac{\tau}{2} \left| \frac{\partial \tilde{A}}{\partial x} \right|^2 \right\}$$

Plugging in the mode expansion

$$\overline{E} = \sum_n \frac{1}{2} M \omega_n^2 |q_n|^2 \qquad M = \rho a$$

Thus, the total energy decomposes into the normal modes. Each mode has ~~an~~ a time-average energy

$$E_n = \frac{1}{2} M \omega_n^2 |q_n|^2$$

If we write $q_n = c_n A \Rightarrow c_n = \frac{q_n}{A}$ (Fraction of amplitude)

$$\Rightarrow E_n = \left(\frac{1}{2} M \omega_n^2 A^2\right) |c_n|^2$$

$$\Rightarrow \boxed{|c_n|^2 = \text{Fraction of the energy in the mode } n}$$

Connection to quantum mechanics: A particle is indivisible. It will be found in one mode or another when measured. However, we may not know in which mode we will find it, only the probability. In lectures 2-3 we related the fraction of outcomes to the probability that that outcome will occur. That is, given an ensemble of "identically prepared particles", if we measure the energy of each the fraction ~~with~~ with energy E_n is $P_n = |c_n|^2$

<u>Completeness:</u> We have considered superpositions of stationary states for the ∞-square well

$$\psi(x) = \sum_{n=1}^{\infty} c_n u_n(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin(nk_1 x)$$

where $k_1 = \frac{\pi}{a}$

This last expression is just the sine <u>Fourier series</u>

Any (reasonably behaved) function on the interval $0 \leq x \leq a$ can be decomposed this way. Given the function $\psi(x)$ such that $\psi(0) = \psi(a) = 0$

then $\psi(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin(nk_1 x)$

where $c_n = \sqrt{\frac{2}{a}} \int_0^a dx \, \sin(nk_1 x) \, \psi(x)$

The set $\left\{ \sqrt{\frac{2}{a}} \sin(nk_1 x) \right\}$ is said to be "<u>complete</u>" since any function can be expressed as a superposition of them.

<u>Initial value problem:</u>

Suppose we are given a wave function at $t=0$ $\psi(x,0)$. We seek the solution to $\frac{\hbar}{-i} \frac{\partial \psi}{\partial t} = \hat{H} \psi$ at $t > 0$

To procedure, we use the fact that the energy eigenfunctions are a complete set

thus $\psi(x,0) = \sum_n c_n u_n(x)$

where $c_n = \int_{-\infty}^{\infty} dx\, u_n^*(x)\, \psi(x,0)$

then $$\boxed{\psi(x,t) = \sum_n c_n\, u_n(x)\, e^{-iE_nt/\hbar}}$$

Thus by decomposing into the normal modes, we immediately solve the initial value problem

Check:

$$\frac{\hbar}{-i}\frac{\partial \psi}{\partial t} = \sum_n c_n u_n \left(\frac{-\hbar}{i}\frac{\partial}{\partial t} e^{-iE_nt/\hbar}\right)$$

$$= \sum_n c_n\, E_n\, u_n(x)$$

$$\hat{H}\psi = \hat{H}\sum_n c_n\, u_n(x)$$

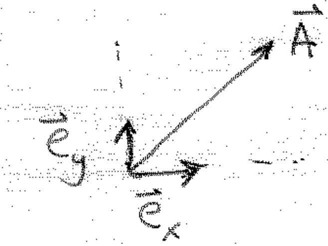
$$= \sum_n c_n \left(\hat{H} u_n(x)\right)$$ since $\hat{H}$ is a linear operator

$$\hat{H} = \sum_n c_n\, E_n\, u_n(x)$$

$$\Rightarrow \frac{\hbar}{-i}\frac{\partial \psi}{\partial t} = \hat{H}\psi \quad \checkmark$$

Vector space analogy:

Consider a vector in two dimensions



$$\vec{A} = c_x\,\vec{e}_x + c_y\,\vec{e}_y$$

$\{\vec{e}_x, \vec{e}_y\}$ = basis of unit vectors

Any vector $\vec{A}$ can be expressed as a linear combination of the basis vectors

$$\Rightarrow \{\vec{e}_x, \vec{e}_y\} = \text{Complete set}$$

To find the expansion coefficients, we can use the dot product.

$$c_x = \vec{e}_x \cdot \vec{A} \quad , \quad c_y = \vec{e}_y \cdot \vec{A}$$

This works because
$$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$$
$$\vec{e}_x \cdot \vec{e}_y = \vec{e}_y \cdot \vec{e}_x = 0$$

the basis is said to be ortho-normal

orthogonal — unit length

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = \text{"Kronecker delta"}$$

The dot product (also known as the inner product) allows us to "pick off the component"

Dirac notation:

We will generally write an inner product as

$$\vec{e}_x \cdot \vec{A} = \underbrace{\langle e_x | A \rangle}_{\text{braket}} \qquad \langle e_x | = \text{"bra"} \qquad |A\rangle = \text{"ket"}$$

MUCH more on this latter on

Hilbert Space: $\mathcal{H}$

Let us consider the expansion of the wave function in terms of energy eigenstates

$$\psi(x) = \sum_n c_n \, u_n(x)$$

Analogy: $\vec{A} = \sum_n c_n \, \vec{e}_n$

$$\{u_n(x)\} = \text{"basis function"} = \text{complete set}$$

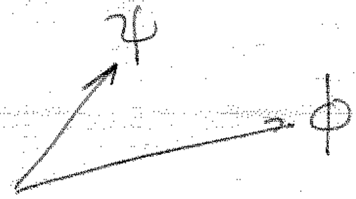
Define Inner product: $\boxed{\langle \psi | \phi \rangle \equiv \int_{-\infty}^{\infty} dx \, \psi^*(x) \, \phi(x)}$

$$c_n = \langle u_n | \psi \rangle = \int_{-\infty}^{\infty} dx \, u_n^*(x) \, \psi(x)$$

"Project out" the expansion coefficient using the inner product.

Thus we define a space $\mathcal{H}$ = Hilbert space.

A function $\psi(x) \in \mathcal{H}$ is like a vector



The space has an inner product

$$\langle \psi | \phi \rangle \equiv \int_{-\infty}^{\infty} dx \, \psi^*(x) \, \phi(x)$$

There are bases = complete set

$$\psi(x) = \sum_n c_n u_n(x)$$

The bases is orthonormal

$$\Rightarrow \langle u_n | u_m \rangle = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

The "norm" of the vector

$$\|\psi\| = \sqrt{\|\psi\|^2} = \sqrt{\langle \psi | \psi \rangle}$$

$$\|\psi\|^2 = \int_{-\infty}^{\infty} dx \, |\psi(x)|^2$$

To be in the Hilbert space we require $\|\psi\| < \infty$. This is an abstract vector space.

Other Bases:

Consider the momentum eigenfunctions

$$\hat{p}\, u_p(x) = p\, u_p(x)$$

Where $u_p(x) = A\, e^{ipx/\hbar}$ (plane wave)

Here p is a continuous variable

If we let p range $-\infty \le p \le +\infty$ the set is complete

$$\psi(x) = \int_{-\infty}^{\infty} dp\, \underset{\substack{\uparrow \\ \text{expansion coefficient}}}{c(p)}\; u_p(x) \leftarrow \text{(basis function)} \qquad \begin{pmatrix}\text{Integral} \\ \text{rather than sum}\end{pmatrix}$$

$$= \int_{-\infty}^{\infty} dp\, c(p)\, A\, e^{ipx/\hbar}$$

Let us choose $A = \frac{1}{\sqrt{2\pi\hbar}} \Rightarrow \psi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, c(p)\, e^{ipx/\hbar}$

But we've seen this before, This is just the Fourier integral. The function

$$c(p) = \tilde{\phi}(p) = \text{the momentum-space wave function}$$

$$\underset{\substack{\uparrow \\ \text{expansion coefficient} \\ \text{by projecting} \\ \text{onto basis vector}}}{c(p) =} \langle u_p | \psi \rangle = \int_{-\infty}^{\infty} dx\, u_p^*(x)\, \psi(x)$$

$$\Rightarrow \tilde{\phi}(p) = c(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \, e^{-ipx/\hbar} \, \psi(x)$$

The moment space wave function thus represents the <u>expansion coefficients</u> in the superposition of momentum eigenfunction

Probability (density) $P(p) = |c(p)|^2 = |\tilde{\phi}(p)|^2$

<u>Orthogonality for continuous variables</u>

$$\langle u_p | u_{p'} \rangle = \int_{-\infty}^{\infty} dx \, u_p^*(x) \, u_{p'}(x)$$

$$= \int_{-\infty}^{\infty} dx \, \frac{e^{-ip\frac{x}{\hbar}} \, e^{ip'x/\hbar}}{2\pi\hbar}$$

Let $y = \frac{x}{\hbar}$ $$= \int_{-\infty}^{\infty} dy \, \frac{e^{i(p'-p)y}}{2\pi}$$

$$\Rightarrow \boxed{\langle u_p | u_{p'} \rangle = \delta(p - p')}$$

Unlike the discrete case, here orthogonality is determined by the <u>Dirac</u> delta

If $p \neq p'$ $\langle u_p | u_{p'} \rangle = 0$

<u>Note</u>: If $p = p'$ $\langle u_p | u_p \rangle = \infty$ <u>Unnormalizable</u>

Position basis:

Position eigenstates: $\hat{x}\, u_{x_0}(x) = x_0\, u_{x_0}(x)$

$\uparrow$ eigenvalue

$$\Rightarrow u_{x_0}(x) = \delta(x - x_0)$$

$$\langle u_{x_1} | u_{x_2} \rangle = \delta(x_1 - x_2) \quad \text{orthogonal}$$

The set $\{u_{x_0}(x)\}$ is complete

if we let x_0 range $-\infty \le x_0 \le +\infty$

$$\Rightarrow \psi(x) = \int_{-\infty}^{\infty} dx_0 \; c(x_0) \; u_{x_0}(x)$$

$c(x_0)$: expansion coefficient; $u_{x_0}(x)$: basis vector

$$= \int_{-\infty}^{\infty} dx_0 \; c(x_0) \; \delta(x - x_0)$$

$$\Rightarrow \psi(x) = c(x) \quad ! \, ?$$

The wave function $\psi(x)$ is thus to be interpreted as the expansion coefficients in the position eigenfunctions

Probability (density) $P(x) = |c(x)|^2 = |\psi(x)|^2$

The Born interpretation!