Lecture 9: The General Structure of Quantum Theory

Up until this point we have built our intuition and machinary to describe the quantum world. We are now in a position to put it all together into a general structure, sometimes known as the "Postulates of Quantum Mechanics"

$\boxed{\text{Postulate 1}}$: Our maximum possible knowledge of a system ("state of system") is described by a vector in Hilbert space, $|\psi\rangle \in \mathcal{H}$

ket

Actually a "ray" in Hilbert space since $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent the same state

Hilbert space: An ∞-Dimensional vector space with complex scalars.

There are different "representations" of $|\psi\rangle$

$$|\psi\rangle$$

$\psi(x)$	$\tilde{\phi}(p)$	$\{c_n\}$
Position rep.	Momentum rep.	Energy rep.

Inner product: $\langle\psi_1|\psi_2\rangle \equiv \int_{-\infty}^{\infty} dx\, \psi_1^*(x)\, \psi_2(x)$

$$= \int_{-\infty}^{\infty} dp\, \tilde{\phi}_1^*(p)\, \tilde{\phi}_2(p) = \sum_n c_n^{(1)*} c_n^{(2)}$$

$\boxed{\text{Postulate 2}}$: The physical "observables" are linear operators $\hat{A}$ on Hilbert space

$$\hat{A}: \mathcal{H} \to \mathcal{H} \quad \text{(map)}$$

Example: Momentum operator $\hat{p}$

Different representations:

$$\hat{p} \begin{cases} \frac{\hbar}{i}\frac{\partial}{\partial x} & \text{Position rep.} \\ p & \text{momentum rep.} \end{cases}$$

$\hat{p}\,|\psi\rangle \to \frac{\hbar}{i}\frac{\partial \psi}{\partial x}$ Position representation

$\quad\quad\ \to p\,\tilde{\Phi}(p)$ Momentum representation

The possible values that can be measured for an observable are its eigenvalues.

$$\underset{\text{operator}}{\hat{A}}\,|\underset{\text{eigenstate}}{a}\rangle = \underset{\text{eigenvalue}}{a}\,|\underset{\text{eigenstate}}{a}\rangle$$

Now, for physical observables, the possible values should be real numbers

An operator with real eigenvalues is said to be a Hermitian operator

$\Rightarrow$ Physical Observable $\Longleftrightarrow$ Hermitian Operator

Aside: Hermitian Operators and Hermitian Adjoint

If an operators is Hermitian, its eigenvalues are real

$\Rightarrow$ Its expectation value is real

$$\Rightarrow \quad \langle \hat{A} \rangle = \langle \hat{A} \rangle^*$$

Now for a given state $|\psi\rangle$

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dx\, \psi^*_{(x)}\, \hat{A}\, \psi(x)$$

$$\Rightarrow \langle \hat{A} \rangle^* = \int_{-\infty}^{\infty} dx\, (\hat{A}\psi_{(x)})^*\, \psi(x)$$

It then follows (see ~~Homework~~ P.S. #4)

$$\boxed{\int_{-\infty}^{\infty} dx\, \phi^*_{(x)}\, \hat{A}\psi(x) = \int_{-\infty}^{\infty} dx\, (\hat{A}\phi(x))^*\, \psi(x) \quad \text{iff } \hat{A} \text{ is Hermitian}}$$

Generally we define the Hermitian Adjoint s.t. given an operator $\hat{A}$, $\hat{A}^\dagger$ (read "$\hat{A}$-dagger") satisfies

$$\int_{-\infty}^{\infty} dx\, \phi^*_{(x)} \left(\hat{A}\psi_{(x)}\right) = \int_{-\infty}^{\infty} dx\, (\hat{A}^\dagger \phi(x))^*\, \psi_{(x)}$$

Hermitian operator $\Rightarrow$ $\hat{A} = \hat{A}^\dagger$ <u>self-adjoint</u>

~~Self~~ adjoint is like complex conjugate for operators

<u>Example</u>: Derivative Operator $\hat{D} \equiv \frac{\partial}{\partial x}$

$$\int_{-\infty}^{\infty} dx\, \phi^*(x)\, \hat{D}\, \psi(x) = \int_{-\infty}^{\infty} dx\, \phi^*(x) \frac{\partial \psi}{\partial x} \underset{\uparrow}{=} \int_{-\infty}^{\infty} dx \left(-\frac{\partial \phi^*}{\partial x}\right) \psi(x)$$

integration by parts

$$\Longrightarrow \hat{D}^\dagger = -\frac{\partial}{\partial x} \Rightarrow \hat{D} \text{ is } \underline{\text{not}} \text{ Hermitian}$$

However $\hat{p} = \frac{\hbar}{i} \hat{D}$ is Hermitian

$$\int_{-\infty}^{\infty} dx\, \phi^*(x)\, \hat{p}\, \psi(x) = \int_{-\infty}^{\infty} dx\, \phi^*_{(x)} \frac{\hbar}{i} \hat{D}\, \psi(x)$$

$$= \frac{\hbar}{i} \int_{-\infty}^{\infty} dx\, \phi^*(x)\, \hat{D}\, \psi(x) = \frac{\hbar}{i} \int_{-\infty}^{\infty} dx\, \left(\hat{D}^\dagger \phi(x)\right)^* \psi(x)$$

$$= \frac{\hbar}{i} \int_{-\infty}^{\infty} dx \left(-\frac{\partial \phi}{\partial x}\right)^* \psi(x) \underset{\uparrow}{=} \int_{-\infty}^{\infty} dx \left(\frac{\hbar}{i} \frac{\partial \phi}{\partial x}\right)^* \psi(x)$$

using $i^* = -i$

$$= \int_{-\infty}^{\infty} dx\, (\hat{p}\, \phi(x))^* \psi(x)$$

$$\Longrightarrow \hat{p} = \hat{p}^\dagger$$

Physical observables are thus self adjoint (Hermitian)

Postulate 3: The Probability of finding measurement outcome "a" for observable $\hat{A}$ given eigenvalue equation $\hat{A}|u_a\rangle = a\,|u_a\rangle$

is

$$\boxed{P_a = |\langle u_a|\Psi\rangle|^2 \quad \text{(Born Rule)}}$$

Note: the eigenstates of $\hat{A}$ form a basis for the Hilbert space

$$\Rightarrow \quad |\Psi\rangle = \sum_a c_a\,|u_a\rangle$$

$$c_a = \langle u_a|\Psi\rangle = \int_{-\infty}^{\infty} dx\, u_a^*(x)\,\Psi(x)$$

"Project out" the a^{th} component in Ψ

- $|\Psi\rangle$ is a quantum <u>superposition</u> of the eigenstates $|u_a\rangle$
- c_a is the "<u>probability amplitude</u>" for a^{th} outcome
- $\boxed{P_a = |c_a|^2 \text{ is the probability of } a^{th} \text{ outcome.}}$

Postulate 4: "Post-Measurement State"

In Lecture 3 we saw "Bayes rule" for classical probabilistic reasoning. Given a probability distribution which describes our state of knowledge, given new information we must update our state assignment.

The same is true quantum mechanically. When we perform a measurement of an observable $\hat{A}$ and find eigenvalue a, then immediately we ~~know~~ make the state assignment $|u_a\rangle$

$$|\psi\rangle \xRightarrow[\text{and find } a]{\text{Measure } \hat{A}} |u_a\rangle$$

pre-measurement state

post-measurement state

this is sometime known as the collapse of the wavefunction

Since before measure $|\psi\rangle = \sum_a c_a |u_a\rangle$ after measurement the state has "collapsed" to one term $|u_a\rangle$.

Aside: Projection Operator

Define $\hat{\mathbb{P}}_a |\Psi\rangle = |u_a\rangle\langle u_a|\Psi\rangle$

$$= |u_a\rangle \int dx\, u_a^*(x)\, \Psi(x)$$

$\hat{\mathbb{P}}_a$ is know as a projection operator

Analogy in 2D vector space $\hat{\mathbb{P}}_x \vec{A} = \vec{e}_x (\vec{e}_x \cdot \vec{A})$

So after a ~~measure~~ measurement

$$|\Psi\rangle \longrightarrow \hat{\mathbb{P}}_a |\Psi\rangle$$ (this state is not normalized)

If we include normalization

$$|\Psi\rangle \rightarrow \frac{\hat{\mathbb{P}}_a |\Psi\rangle}{\| \hat{\mathbb{P}}_a |\Psi\rangle \|} = \frac{|u_a\rangle\langle u_a|\Psi\rangle}{|\langle u_a|\Psi\rangle|} = |u_a\rangle$$

as before

The collapse of the wave function is know as the von Neumann projection after the great Mathematician John von Neumann who formulated this theory.

Postulate 5: Quantum Dynamics

If we do not measure the system, then we must update our system according to the time dependent Schrödinger equation.

Given $|\psi(0)\rangle$, then $|\psi(t)\rangle$ satisfies

$$\boxed{\frac{\hbar}{-i}\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$$

$\hat{H}$ is the Hamiltonian, which determines the energies of the system.

All the physical law is determined by the Hamiltonian:

the eigenfunctions of $\hat{H}$ are the "stationary states"

$$\boxed{\hat{H}|u_E\rangle = E|u_E\rangle}$$

Time Independent Schrödinger Equation

Given $|\psi(0)\rangle = \sum_E c_E |u_E\rangle$

$$|\psi(t)\rangle = \sum_E c_E e^{-iEt/\hbar} |u_E\rangle$$

The issue of measurement:

We have seen that the state of the system changes in two distinct ways

(i) Deterministic Schrödinger evolution

$$|\psi(0)\rangle \longrightarrow |\psi(t)\rangle \quad \text{where} \quad \frac{\hbar}{-i}\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

(ii) Stochastic collapse of the wave function

$$|\psi(0)\rangle \rightarrow |u_a\rangle \quad \text{with probability} \quad P_a = |\langle u_a|\psi(0)\rangle|^2$$

What distinguishes them is measurement of the quantum system. But what constitutes a measurement? Isn't a measurement another physical interaction with a measuring device? Isn't there a Hamiltonian that describes everything, including the measuring apparatus?

This has been one of the major sticking points in the quantum theory and has led to volumes of debate on the subject. In the "Copenhagen Interpretation" (a'la Bohr), the world is divided into two parts "quantum" and "classical". The measuring device is to be considered in the classical world. It is characterized by an irreversible measurement record, e.g. a photon is absorbed and amplifies a spot on a photographic film.

This is all very "fuzzy". Where exactly does this quantum/classical boundary lie? What characterizes it?

Today we have answered many of these questions. Bohr, in his genius and deep insight, essentially got it right. A measuring device is a macroscopic object with many degrees of freedom. This complicated object acts irreversibly (as in thermodynamics). We have a theory which describes this quantum irreversible behavior. It leads to decoherence – the removal of quantum interference, replaced instead by classical alternatives.

More on this next semester...