Lecture 10: Compatible Observables

In lecture we saw the uncertainty principle (a'la Heisenberg)

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

which followed from Fourier duality, $\Delta x \Delta k \geq \frac{1}{2}$,
where Δx is the standard deviation of $|\psi(x)|^2$
and Δk " " " " of $|\tilde{\phi}(k)|^2$

Physically, this means that there are no possible states in nature (if in principle) for which the value of x and p are simultaneously known with perfect certainty. Any attempt to determine x will necessarily result in a post-measurement state with a highly uncertain p

Von Neuman projection: Measure x, find x_0

$$\underset{\text{pre-measurement state}}{\psi_-(x)} \Longrightarrow \psi_+(x) = \delta(x - x_0) \leftarrow \text{post-measurement state}$$

But, the momentum wave function after the measurement $\tilde{\phi}_+(p) = \dfrac{e^{-ipx_0/\hbar}}{\sqrt{2\pi\hbar}} \Rightarrow |\tilde{\phi}_+(p)|^2 = \dfrac{1}{2\pi\hbar}$

$$\Longrightarrow \Delta p = \infty \quad \text{Completely uncertain.}$$

The Heisenberg uncertainty principle is an example of a broader class of uncertainty relations. The observables $\hat{p}$ and $\hat{x}$ are said to be <u>incompatible</u>. That is, they have <u>no common eigenstates</u>. Thus, there is no measurement one can do which will ~~yei~~ yield a definite value, with probability 1, for <u>both</u> x and p.

<u>The Commutator</u>:

An important algebraic relation on operators is the so called commutator:

$$\hat{C} \equiv [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

If $\hat{C} = 0$, $\hat{A}$ and $\hat{B}$ are said to <u>commute</u>.

Ordinary numbers commute $5 \times 3 - 3 \times 5 = 0$

If $\hat{C} \neq 0$ then $\hat{A}\hat{B} \neq \hat{B}\hat{A}$

Consider $[\hat{x}, \hat{p}]$. The calculate $\hat{C}$ we find ~~out~~ how it ~~operatores~~ operates on vectors in Hilbert space.

$$\hat{x}\hat{p}|\psi\rangle \doteq x\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x) = \frac{\hbar}{i}\, x \frac{\partial \psi}{\partial x}$$

represented in position space

$$\hat{p}\hat{x}|\psi\rangle \doteq \frac{\hbar}{i}\frac{\partial}{\partial x}\left(x\psi(x)\right) = \frac{\hbar}{i}x\frac{\partial\psi}{\partial x} + \frac{\hbar}{i}\psi(x)$$

$$\Rightarrow [\hat{x},\hat{p}]|\psi\rangle = (\hat{x}\hat{p}-\hat{p}\hat{x})|\psi\rangle = -\underbrace{\frac{\hbar}{i}}_{i\hbar}|\psi\rangle$$

$$\rightarrow \boxed{[\hat{x},\hat{p}] = i\hbar}$$

This is a fundamental relation in quantum physics, sometimes known as the "canonical commutation relation"

Some basics of commutation algebra

- Any operator commutes with itself

$$[\hat{A},\hat{A}] = \hat{A}^2 - \hat{A}^2 = 0$$

- Any operator commutes with a function of itself

$$[\hat{A}, f(\hat{A})] = 0$$

Aside: function of an operator - use Taylor series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{where} \quad c_n = \frac{1}{n!}\frac{d^nf}{dx^n}\bigg|_{x=0}$$

$$\Rightarrow f(\hat{A}) \equiv \sum_{n=0}^{\infty} c_n \hat{A}^n$$

- $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ obvious

- $[\hat{A}, c_1\hat{B}_1 + c_2\hat{B}_2] = c_1[\hat{A}, \hat{B}_1] + c_2[\hat{A}, \hat{B}_2]$

$\Uparrow$ linear operators

- $[\hat{A}, \hat{B}_1\hat{B}_2] = [\hat{A}, \hat{B}_1]\hat{B}_2 + \hat{B}_1[\hat{A}, \hat{B}_2]$

$\Uparrow$ like product rule for derivatives

Check: $[\hat{A}, \hat{B}_1\hat{B}_2] = \hat{A}\hat{B}_1\hat{B}_2 - \hat{B}_1\hat{B}_2\hat{A}$

$[\hat{A}, \hat{B}_1]\hat{B}_2 = (\hat{A}\hat{B}_1 - \hat{B}_1\hat{A})\hat{B}_2 = \hat{A}\hat{B}_1\hat{B}_2 - \hat{B}_1\hat{A}\hat{B}_2$

$\hat{B}_1[\hat{A}, \hat{B}_2] = \hat{B}_1(\hat{A}\hat{B}_2 - \hat{B}_2\hat{A}) = \hat{B}_1\hat{A}\hat{B}_2 - \hat{B}_1\hat{B}_2\hat{A}$

add $= \hat{A}\hat{B}_1\hat{B}_2 - \hat{B}_1\hat{B}_2\hat{A}$ ✓

Commutation algebra is an important piece of machinery we will employ in the future in calculating properties of quantum systems

Commutation Relations and Simultaneous Eigenvectors

Suppose there is a complete set of vectors (i.e. a basis $\{|u_{ab}\rangle\}$ such that

$$\hat{A}|u_{ab}\rangle = a|u_{ab}\rangle$$
$$\hat{B}|u_{ab}\rangle = b|u_{ab}\rangle$$

The kets $\{|u_{ab}\rangle\}$ are simultaneously eigenvectors of $\hat{A}$ and $\hat{B}$

Since any vector can be expanded in this basis

$$|\psi\rangle = \sum c_{ab} |u_{ab}\rangle$$

we have

$$\hat{A}\hat{B} |\psi\rangle = \sum c_{ab} \hat{A}\hat{B} |u_{ab}\rangle = \sum c_{ab}\, ab\, |u_{ab}\rangle$$

$$\hat{B}\hat{A} |\psi\rangle = \sum c_{ab} \hat{B}\hat{A} |u_{ab}\rangle = \sum c_{ab}\, ba\, |u_{ab}\rangle$$

but $ab = ba$ (numbers commute)

$$\Rightarrow \quad \hat{A}\hat{B} = \hat{B}\hat{A} \quad \Rightarrow \quad [\hat{A},\hat{B}] = 0$$

Thus if $\{|u_{ab}\rangle\}$ is a complete set of eigenvectors of two operators $\hat{A}$ and $\hat{B}$, then $\hat{A}$ and $\hat{B}$ commute

The converse is also true

If $[\hat{A},\hat{B}] = 0$ then there exists a ~~[illegible]~~ complete set of vectors which are simultaneously eigenvectors of $\hat{A}$ and of $\hat{B}$

To see this, suppose $[\hat{A},\hat{B}] = 0$

Suppose $\hat{A}|u_a\rangle = a|u_a\rangle$ (eigenvector equation)

and $|u_a\rangle$ is <u>non-degenerate</u>

(i.e. unique up to normalization)

Let $|\phi_a\rangle \equiv \hat{B}|u_a\rangle$

the $\hat{A}|\phi_a\rangle = \hat{A}\hat{B}|u_a\rangle = \hat{B}\hat{A}|u_a\rangle = a\hat{B}|u_a\rangle$

$$= a|\phi_a\rangle$$

$\Rightarrow |\phi_a\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue a

But since the eigenvalue is <u>not degenerate</u> we must have $|\phi_a\rangle \propto |u_a\rangle$

$\Rightarrow |\phi_a\rangle = \hat{B}|u_a\rangle = b|u_a\rangle$

$\therefore |u_a\rangle$ is an eigenvector of $\hat{B}$ with eigenvalue b. So now we label the ket with both eigenvalues $|u_{ab}\rangle$

The case of degeneracy is more complicated. We'll touch on it soon, but return to a more detailed discussion later

<u>Simultaneous eigenvectors</u> $\Rightarrow$ One can measure both ~~and~~ $\hat{A}$ and $\hat{B}$ and get definite values, with probability 1 $\Rightarrow$ No uncertainty in either $\hat{A}$ or $\hat{B}$. Thus there is an intimate relationship between uncertainty relations and commutation relations

General Uncertainty Relations

Consider two observables described by operators $\hat{A}$ and $\hat{B}$. It the follows that

$$\boxed{\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|}$$

where $\langle [\hat{A}, \hat{B}] \rangle$ = expectation value of $\hat{A}\hat{B} - \hat{B}\hat{A}$ in a state of interest

$$\Delta A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \qquad \Delta B^2 = \langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2$$

For example: $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}$

$$\Rightarrow \Delta x \, \Delta p \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{p}] \rangle \right| = \frac{1}{2} \left| \langle \overset{\text{constant}}{i\hbar} \rangle \right| = \frac{|i\hbar|}{2}$$

$$\Rightarrow \boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$ Heisenberg uncertainty principle

The proof of the general uncertainty relation will have to wait for a later lecture when we have more mathematical machinery.

The basic point is clear. If two operators commute they have simultaneous eigenvectors. Then it is in principle possible to have both ΔA and $\Delta B = 0$

$\Rightarrow$ Compatible operators.

Conservation laws

Consider a quantum system with Hamiltonian $\hat{H}$, so that $\frac{\hbar}{-i}\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$

The Hamiltonian determines the time evolution (i.e. the dynamics).

Suppose there is an operator which commutes with $\hat{H}$, $[\hat{H}, \hat{A}] = 0$

$\Rightarrow$ $\hat{A}$ is <u>conserved with time</u>

~~That~~ is, there exist simultaneous eigenvectors of $\hat{H}$ and $\hat{A}$. But the eigenvectors of $\hat{H}$ are <u>stationary states</u>, so these states remain eigenvectors of $\hat{A}$ for all times.

<u>For example:</u> Consider a free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$

Then momentum is conserved, $[\hat{H}, \hat{p}] = [\frac{\hat{p}^2}{2m}, \hat{p}] = 0$

There exist simultaneous eigenstate of $\hat{H}$ and $\hat{p}$,

these are plane waves: $u_p(x) = \frac{1}{\sqrt{2\pi}} e^{ipx/\hbar}$

$$\hat{H}\, u_p(x) = \frac{p^2}{2m}\, u_p(x) \qquad \hat{p}\, u_p(x) = p\, u_p(x)$$