Physics 491: Lecture 10b

Beyond Projective Measurement

According to the fundamental measurement postulate, when measuring an observable $\hat{A}$, which corresponds to a Hermitian operator, the measurement outcomes are labeled by the real eigenvalues $\{a\}$ and corresponding eigenvectors $\hat{A}|u_a\rangle = a|u_a\rangle$. Given a state assignment $|\psi\rangle$, outcome "a" occurs with probability $P_a = |\langle u_a|\psi\rangle|^2$.

The post-measurement state

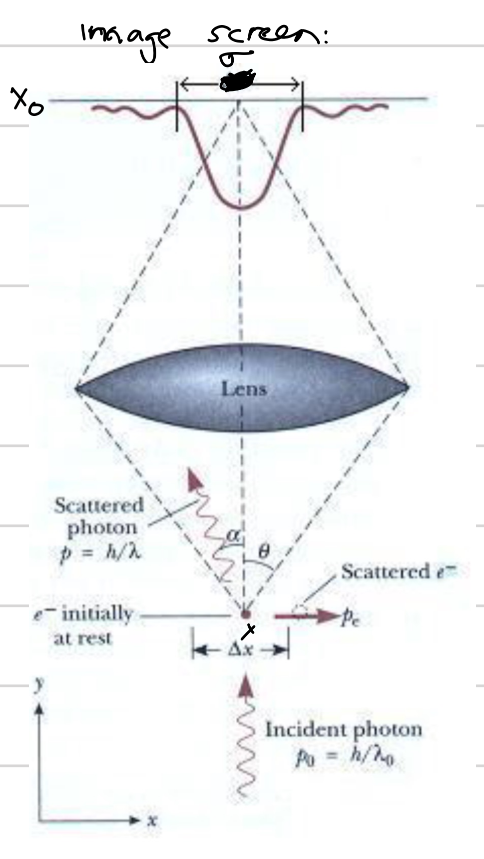
$$|\psi\rangle\Big|_a = |u_a\rangle = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\|\hat{\mathbb{P}}_a|\psi\rangle\|}$$

where $\hat{\mathbb{P}}_a = |u_a\rangle\langle u_a|$ is the "projection operator.

Note $\|\hat{P}_a|\psi\rangle\|^2 = \||\hat{P}_a\psi\rangle\|^2 = \langle \hat{P}_a\psi|\hat{P}_a\psi\rangle = \langle\psi|\hat{P}_a^\dagger\hat{P}_a|\psi\rangle$

$$= \langle\psi|\hat{P}_a\hat{P}_a|\psi\rangle = \langle\psi|\hat{P}_a|\psi\rangle = \langle\psi|u_a\rangle\langle u_a|\psi\rangle = |\langle u_a|\psi\rangle|^2$$

However, in reality, not every measurement is projective. For example, as we have already seen, in Heisenberg microscope, we do not exactly measure x, but only learn x within the finite resolution of the microscope, σ



In a classical imaging system, we can define the "likelihood" function to find x_0 on the image screen, given the particle was at x

$$P_{LH}(x_0|x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_0-x)^2}{2\sigma^2}} \quad \text{(Approximate as Gaussian)}$$

Then, if $P_{in}(x)$ is the "prior probability" assignment, then according to Bayes' rule, out posteria "output probability is

$$P_{out}(x|x_0) = \frac{P_{LH}(x_0|x)\,P_{in}(x)}{\int dx\; P_L(x_0|x)\,P_{in}(x)}$$

We write Bayes rule: $P_{out}(x|x_0) = \dfrac{\tilde{P}_{out}(x|x_0)}{\int_{-\infty}^{\infty} dx \, \tilde{P}_{out}(x|x_0)}$

where $\tilde{P}_{out}(x|x_0) = P_{LH}(x|x_0) \, P_{in}(x_0)$ in the "unnormalized Posterior"

For example, suppose the prior is itself Gaussian

$$P_{in}(x) = \frac{1}{\sqrt{2\pi \Delta x^2}} e^{-\frac{(x-\langle x \rangle_{in})^2}{2\Delta x_{in}^2}}$$

Then as we have seen: $P_{out}(x|x_0) = \dfrac{1}{\sqrt{2\pi \Delta x_{out}^2}} \exp\left\{-\dfrac{(x-\langle x \rangle_{out})^2}{2\Delta x_{out}^2}\right\}$

$$\Delta x_{out}^2 = \frac{\Delta x_{in}^2}{1+r}, \qquad \langle x \rangle_{out} = \langle x_{in} \rangle + \frac{r}{1+r}(x_0 - \langle x \rangle_{in})$$

where $r = \dfrac{\Delta x_{in}^2}{\sigma^2}$ ("measurement strength")

When $\sigma \to 0$ $\quad r \to \infty \quad \langle x \rangle_{out} \to x_0, \quad \Delta x_{out}^2 \to 0$: $P_{out}(x|x_0) = \delta(x-x_0)$
Projective!

When $\sigma \to \infty$ $\quad r \to 0 \quad \langle x_{out} \rangle = \langle x_{in} \rangle, \quad \Delta x_{out}^2 = \Delta x_{in}^2$: No info
$\Rightarrow$ No update

In Quantum mechanics, we can generalize beyond von Neumann projective measurements using a set of "measurement" operators

For example, consider the Heisenberg microscope. Let us define

$$\hat{K}_{x_0} \equiv \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left\{-\frac{(\hat{x}-x_0)^2}{4\sigma^2}\right\}, \quad \text{(Measurement (Kraus) operator)}$$

Measurement rule: Given input state $|\psi_{in}\rangle$

Probability density of measurement outcome x_0

$$P(x_0) \equiv \| \hat{K}_{x_0} |\psi_{in}\rangle \|^2 = \langle \psi_{in} | \hat{K}_{x_0}^{\dagger} \hat{K}_{x_0} |\psi_{in}\rangle$$

Post-measurement state: $|\psi_{out}\rangle = \dfrac{\hat{K}_{x_0} |\psi_{in}\rangle}{\| \hat{K}_{x_0} |\psi\rangle \|}$

Quantum Bayes Rule

As $\sigma \to 0$ $\quad \hat{K}_{x_0} \to \delta(\hat{X} - x_0) = |x_0\rangle\langle x_0|$ Projective measurement
$= \hat{P}_{x_0}$

As $\sigma \to \infty$ "Weak measurement" (No disturbance)

Suppose, for example, we initial prepare the particle in a Gaussian wave packet

$$\psi_{in}(x) = \langle x|\psi_{in}\rangle = \frac{1}{(2\pi \Delta x_{in}^2)^{1/4}} \exp\left\{ -\frac{(x - \langle x_{in}\rangle)^2}{4\Delta x_{in}^2} \right\}$$

$$\Rightarrow \psi_{out}(x) = \frac{\langle x|\hat{K}_{x_0}|\psi_{in}\rangle}{\|\hat{K}_{x_0}|\psi_{in}\rangle\|} = \frac{1}{(2\pi \Delta x_{out}^2)^{1/4}} e^{-\frac{(x - \langle x_{out}\rangle)^2}{4\Delta x_{out}^2}}$$

$$P_{out}(x) = |\psi_{out}(x)|^2 = \frac{1}{\sqrt{2\pi \Delta x_{out}^2}} e^{-\frac{(x - \langle x_{out}\rangle)^2}{2\Delta x_{out}^2}}$$

Posterior probability density, just as in Bayes rule!

So, this part of the "collapse of the wave function" is not surprising that – it's just Baysian updating! Of course, in the classical world this measurement outcome is seen to be a "pre-existing element of reality" where as in quantum mechanics we really can't say that, but never mind that.

So what's different in quantum mechanics? The "reduction of the wave function" in Baysean. But measurement backaction implies that a measurement of x must affect p! This is completely not classical. The output momentum space wave function is

$$|\tilde{\Phi}_{out}(p)|^2 = \frac{1}{\sqrt{2\pi \Delta p_{out}^2}} e^{-\frac{p^2}{2\Delta p_{out}^2}} \qquad \Delta p_{out} = \frac{1}{2}\frac{\hbar}{\Delta x_{out}}$$

The more we learn about x, the less certain we become of p!