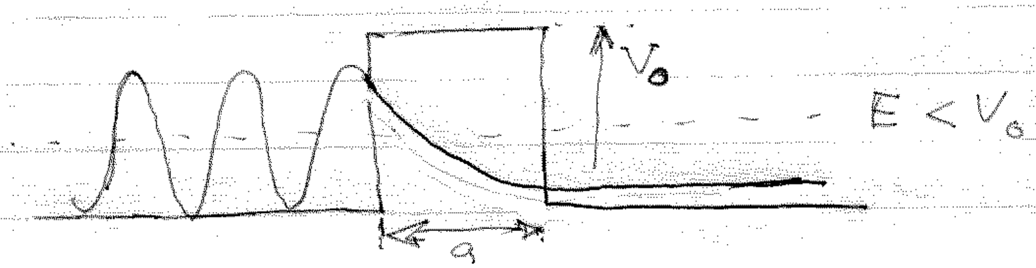
Lecture 15: Tunnelling

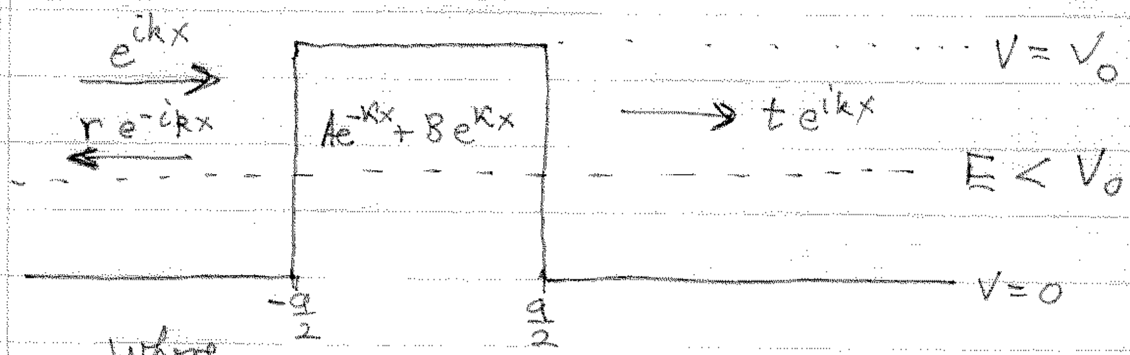
Last lecture we saw that a particle incident on a potential step at an energy smaller than V_0 has a finite probability of penetrating into the classically forbidden region. Ultimately, in this case, the particle is reflected with unit probability. A more startling contradiction with classical intuition is the case of a potential barrier



Sketched above is the square of a wave function for which $E < V_0$ with an incident wave from the left. A good fraction of the wave is reflected. However, some of the wave penetrates the barrier and emerges as a propagating wave on the other side. This phenomenon is known as tunnelling, and is one of the strangest quantum mechanical phenomenon.

(Next Page)

Solving for the transmission probability:



where

$$k=\sqrt{\frac{2mE}{\hbar^2}}\,,\qquad \kappa=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

Note: As the overall amplitude of the wave is arbitrary, I choose the incident wave to have unit amplitude. The reflected wave has amplitude r and transmitted t

Conservation of probability: $|r|^2+|t|^2=1$

This together with the boundary conditions:
$\frac{1}{u}\frac{du}{dx}$ continuous yield r and t

$\underline{\underline{x=-\frac{a}{2}}}$
$$\frac{ik\left(e^{-\frac{ika}{2}}-re^{\frac{ika}{2}}\right)}{e^{-\frac{ika}{2}}+re^{\frac{ika}{2}}}=\frac{-\kappa\left(Ae^{\frac{\kappa a}{2}}-Be^{-\kappa a/2}\right)}{Ae^{\frac{\kappa a}{2}}+Be^{-\kappa a/2}}$$

$\underline{\underline{x=+\frac{a}{2}}}$
$$\frac{-\kappa\left(Ae^{-\frac{\kappa a}{2}}-Be^{\kappa a/2}\right)}{Ae^{-\frac{\kappa a}{2}}+Be^{\kappa a/2}}=\frac{ikte^{ika/2}}{te^{ika}}=ik$$

These equations are messy, but can be used to eliminate A and B

After tedious algebra:

$$t = e^{-ika} \frac{2k\kappa}{2k\kappa \cosh(\kappa a) - i(k^2-\kappa^2)\sinh(\kappa a)}$$

$\Rightarrow$ Transmission coefficient

$$T = |t|^2 = \frac{(2k\kappa)^2}{(k^2-\kappa^2)\sinh^2(\kappa a) + (2k\kappa)^2\cosh^2(\kappa a)}$$

$$\Rightarrow \boxed{T = \frac{(2k\kappa)^2}{(k^2+\kappa^2)^2\sinh^2(\kappa a) + (2k\kappa)^2}}$$

$$k^2 = \frac{2m}{\hbar^2}E \qquad \kappa^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\Rightarrow \boxed{T = \frac{4\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}{\sinh^2(\kappa a) + 4\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}}$$

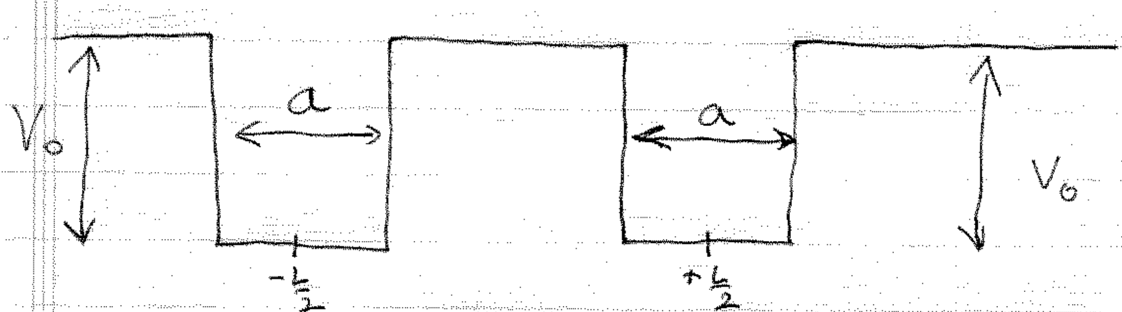
Note: $\kappa a \gg 1 \Rightarrow \sinh(\kappa a) \to \frac{e^{\kappa a}}{2} \gg 1$

$$\Rightarrow \boxed{T \simeq 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-\kappa a}}$$

Exponentially sensitive to barrier via κa

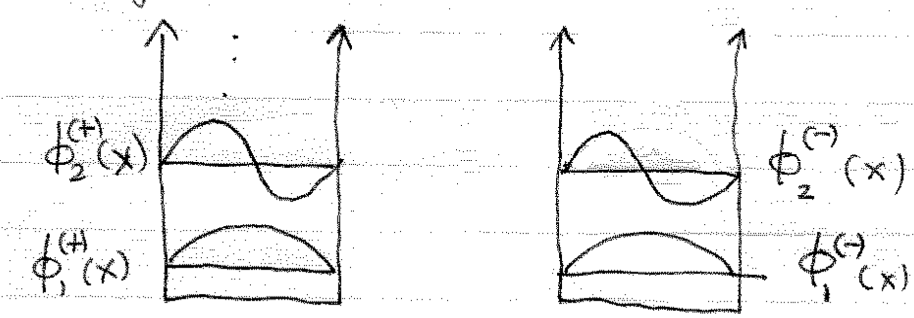
The Double Well and Tunneling Oscillations

considered a "two - box" potential (sometimes known as a "double - well")



When $V_0 \to \infty$. The eigenstates were doubly degenerate, and localized $\{\phi_n^{(\pm)}(x) = u_n(x \pm \frac{L}{2})\}$

$E_n = n^2 \frac{\pi^2 \hbar^2}{2ma^2}$

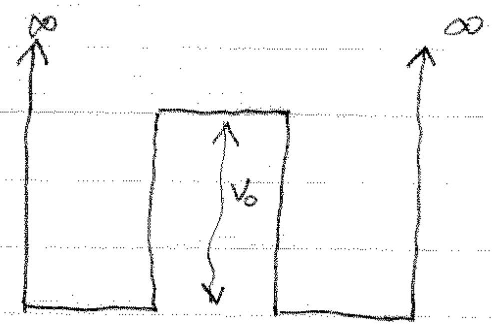


The sets $\{\phi_n^{(+)}\}$ and $\{\phi_n^{(-)}\}$ are orthogonal (they don't overlap). Thus, we saw that if the particle started localized in, e.g., the left box, it stayed there forever, with no probability to be found in the right box.

$$\Psi_n(x,t) = \sum_n \left(c_n^{(+)} \phi_n^{(+)}(x) + c_n^{(-)} \phi_n^{(-)}(x)\right) e^{-iE_nt/\hbar}$$

$$P_{Left} = \sum_n |c_n^{(+)}|^2 \quad , \quad P_{Right} = \sum_n |c_n^{(-)}|^2$$

Now suppose the barrier between the wells is finite



A particle, initially localized in the left box can now tunnel into the right box. Thus, the states localized on either sider are not stationary states.

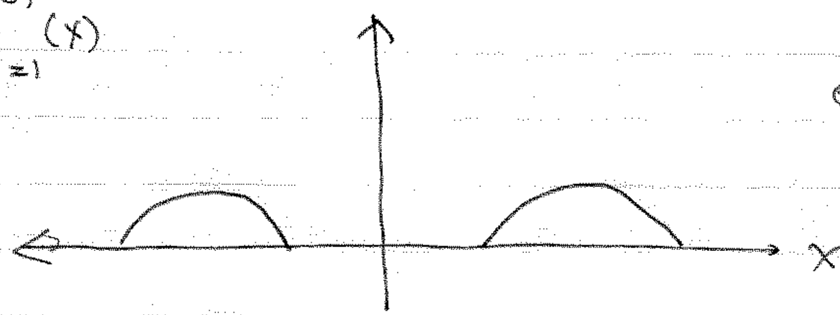
We can approximate the stationary states by considering symmetry. Unlike the two infinite wells, with the finite barrier there are no degeneracies. Because the potential is reflection symmetric, the energy eigenfunctions must be eigenstates of parity.

For infinite wells, $\Phi_n(x) = c_1\, \phi_n^{(+)}(x) + c_2\, \phi_n^{(-)}(x)$ are are stationary states with eignvalue E_n

Define: $$\Phi_n^{(e)}(x) = \frac{\phi_n^{(+)}(x) + \phi_n^{(-)}(x)}{\sqrt{2}}$$

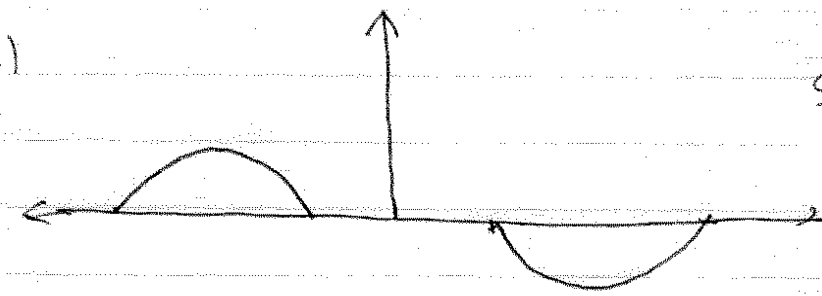
$$\Phi_n^{(o)}(x) = \frac{\phi_n^{(+)}(x) - \phi_n^{(-)}(x)}{\sqrt{2}}$$

Eg. $\Phi^{(e)}_{n=1}(x)$



even parity

$\Phi^{(o)}_{n=1}(x)$

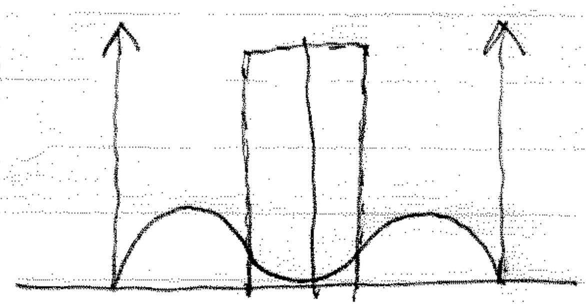


odd parity

$\Phi^{(e)}_{n} = \Phi^{(e)}_{n}(x)$, $\Phi^{(o)}_{n} = -\Phi^{(o)}_{n}(x)$

For the finite case, these are no longer degenerate, because wave functions differ in the classically forbidden region

$\Phi^{(e)}_{n=1} \rightarrow \Phi^{(sym)}_{n=1}(x)$

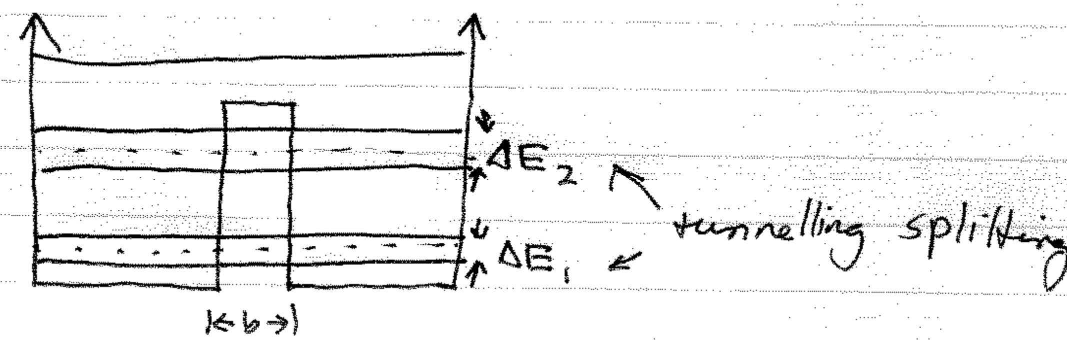


ground state

$\Phi^{(o)}_{n=1} \rightarrow \Phi^{(anti)}_{n=1}(x)$

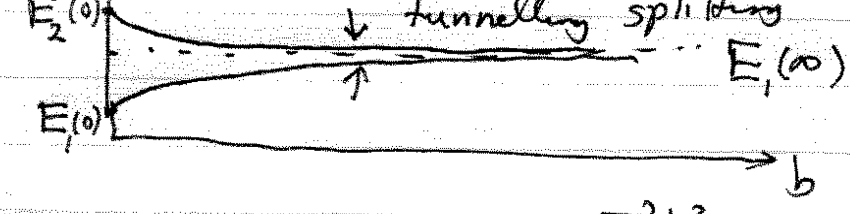
first excited state

The difference in energy between $\Phi_n^{(sym)}$ and $\Phi_n^{(anti)}$ is due to tunnelling. For $\kappa a \gg 1$, we expect this splitting to be small.



<u>Note</u>: I have sketched a situation here such that there are only two bound states below-barrier. The dotted lines represent the energy levels when $b \to \infty$ (i.e. no tunnelling)

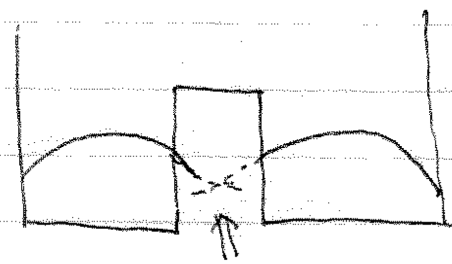
To better understand this behavior, consider a sketch of the energy levels as a function of b



where $E_n(0) = n^2 \dfrac{\pi^2 \hbar^2}{2m(2a+b)^2}$

$E_n(\infty)$ = Bound states of half-infinite, half-finite well

It is useful to think of a kind of "interaction" between the wave function is the left well and that in the right (we'll formalize this next semester)



overlaping tails $\Longrightarrow$ perturbation to energy levels

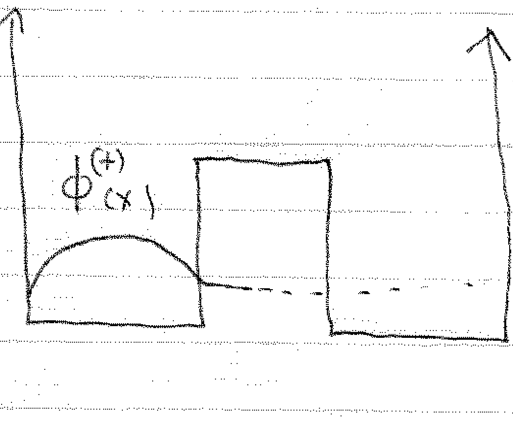
Thus, in some approximation $\Delta E_{\text{tunnelling}} \propto \left|\langle \psi_{\text{left}} | \psi_{\text{right}} \rangle\right|^2$

<u>Tunnelling oscillations</u>:

Consider an initial preparation of a state

$$\psi(x,0) = \phi^{(+)}(x) = \text{state localized on left}$$

$$= \frac{1}{\sqrt{2}}\left(\Phi_1^{(e)}(x) + \Phi_1^{(o)}(x) \right)$$



This is not a <u>stationary state</u>; it is a superposition of energy eigenstates with different energy

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(\Phi_1^{(e)}(x)\, e^{-i\frac{E_1^{(e)}}{\hbar}t} + \Phi_1^{(o)}(x)\, e^{-iE_1^{(o)}t/\hbar}\right)$$

$$= \frac{e^{-iE_1^{(e)}\frac{t}{\hbar}}}{\sqrt{2}}\left(\Phi_1^{(e)}(x) + \Phi_1^{(o)}(x)\, e^{-i\Delta E_1 t/\hbar}\right)$$

neglect

where $\Delta E_1 = E_1^{(o)} - E_1^{(e)}$ (tunnelling splitting)

$$\Rightarrow \psi(\quad) = \frac{1}{\sqrt{2}}\left[\frac{\phi_1^{(+)}(x) + \phi_1^{(-)}(x)}{\sqrt{2}} + \left(\frac{\phi_1^{(+)}(x) - \phi_1^{(-)}(x)}{\sqrt{2}}\right) e^{-i\frac{\Delta E_1 t}{\hbar}}\right]$$

$$\psi(x,t) = \phi_1^{(+)}(x)\left(\frac{1+e^{-i\frac{\Delta E_1 t}{\hbar}}}{2}\right) + \phi_1^{(-)}(x)\left(\frac{1-e^{-i\frac{\Delta E_1 t}{\hbar}}}{2}\right)$$

$$= \cancel{e^{-i\frac{\Delta E_1 t}{2\hbar}}}\left[\phi_1^{(+)}(x)\left(\frac{e^{i\Delta E_1 t/2\hbar} + e^{-i\Delta E_1 t/2\hbar}}{2}\right) + \phi_1^{(-)}(x)\left(\frac{e^{i\Delta E_1 t/2\hbar} - e^{-i\Delta E_1 t/2\hbar}}{2i}\right)\right]$$

neglect

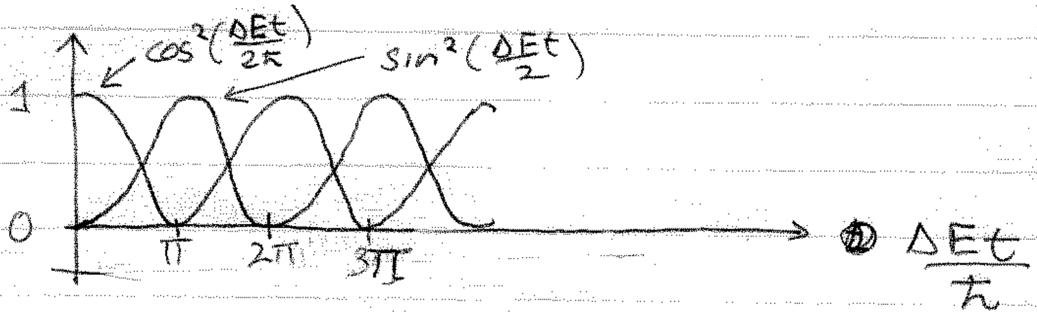
$$\Rightarrow \boxed{\psi(x,t) = \phi_1^{(+)}(x)\cos\left(\frac{\Delta E_1 t}{2\hbar}\right) + \phi_1^{(-)}(x)\sin\left(\frac{\Delta E_1 t}{2\hbar}\right)}$$

(Next Page)

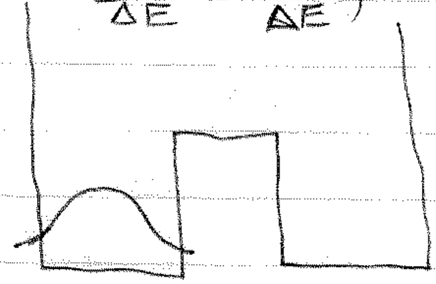
If we approximate $\phi^{(+)}(x)\,\phi^{(-)}(x) \approx 0$
(neglect overlap)

$$\boxed{|\Psi(x,t)|^2 \approx |\phi_1^{(+)}(x)|^2 \cos^2\left(\frac{\Delta E t}{2\hbar}\right) + |\phi_1^{(-)}(x)|^2 \sin^2\left(\frac{\Delta E t}{2\hbar}\right)}$$

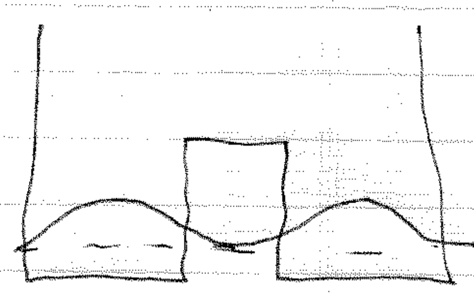
~~Ths~~ Aside: $\cos^2(\theta) = \frac{1+\cos 2\theta}{2}$ $\quad\quad \sin^2\theta = \frac{1-\sin 2\theta}{2}$



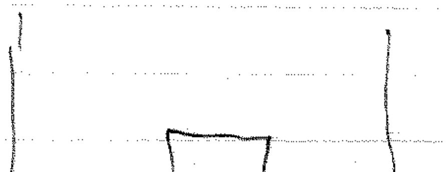
Thus we see that the probability to find the particle in the right of left well oscillates at the frequency $\frac{\Delta E}{\hbar}$, period $\left(T = \frac{2\pi\hbar}{\Delta E} = \frac{h}{\Delta E}\right)$



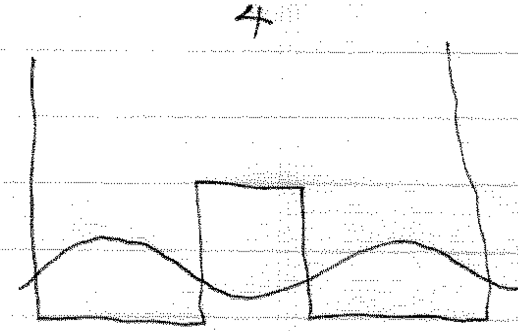
$t = 0$



$t = \frac{T}{4}$



$t = \frac{T}{2}$



$t = \frac{3T}{4}$