Physics 491: Quantum Mechanics I

Simple Harmonic Oscillator: Solution to the T.I.S.E.

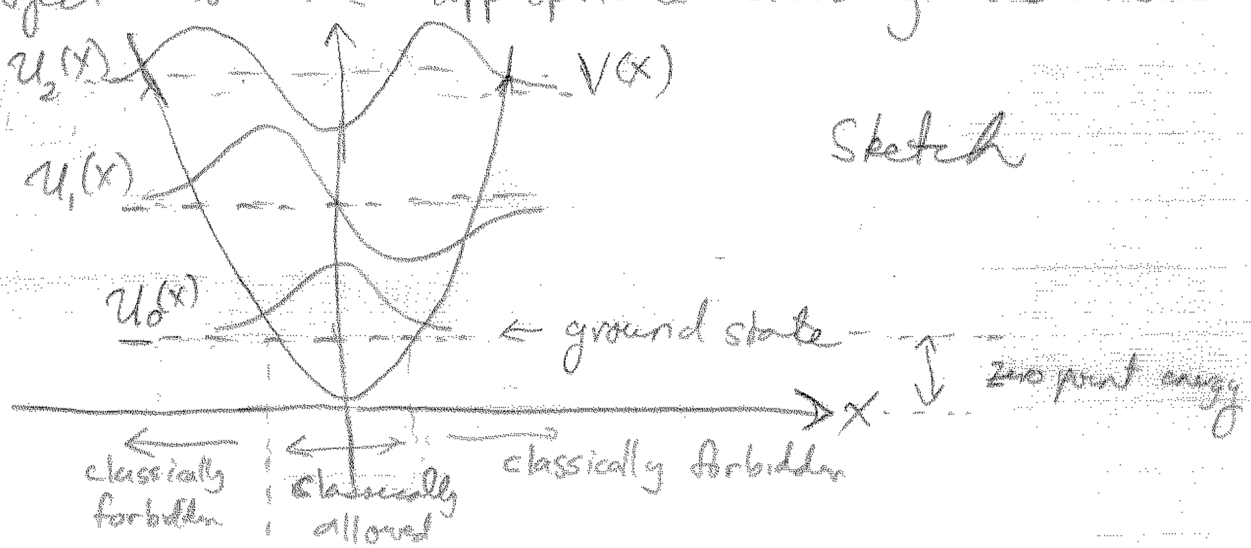
We seek solutions to the Time Independent Schrödinger Equation for the Simple Harmonic Oscillator.

$$\hat{H}\,|u_E\rangle = E\,|u_E\rangle$$

where $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

One approach is to solve the differential equation $\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right) u(x) = E\, u(x)$

subject to the appropriate boundary conditions



From our studies last semester we can sketch the energy eigenfunctions. Note $V(x) = V(-x)$

$\Rightarrow$ Energy eigenfunctions are parity eigenfunctions and alternate even/odd. Each excited state has one more node.

Like the infinite square well, the harmonic well is infinitely high

$\Rightarrow$ Only bound state

$\Rightarrow$ Spectrum of $\hat{H}$ is discrete (non-degenerate)

$\Rightarrow$ Hilbert space $\mathcal{L}_2(\mathbb{R})$ spanned by eigenfunctions of $\hat{H}$

Instead of using differential equation we can use the algebraic methods.

Let us write $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$

where $\hat{N} = \hat{a}^{+}\hat{a}$

Recall $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^{+}] = \hat{a}^{+}$

$\hat{N}$ is a "positive operator" $\Rightarrow$ eigenvalues are ≥ 0

Let $\hat{N}|u_\lambda\rangle = \lambda|u_\lambda\rangle \qquad \lambda \geq 0$

$$\Rightarrow \hat{H}|u_\lambda\rangle = \underbrace{\hbar\omega(\lambda + \tfrac{1}{2})}_{E_\lambda}|u_\lambda\rangle$$

We thus seek λ

__Lemma:__ Let $|v_\lambda\rangle = \hat{a}|u_\lambda\rangle$

$$\Rightarrow \hat{N}|v_\lambda\rangle = (\lambda-1)\,|u_\lambda\rangle$$

i.e. $\hat{a}|u_\lambda\rangle$ eigenvector of $\hat{N}$, eigenvalue $\lambda-1$

Proof: $\hat{N}|v_\lambda\rangle = \hat{N}\hat{a}|u_\lambda\rangle$

$$= (\hat{a}\hat{N} + [\hat{N},\hat{a}])\,|u_\lambda\rangle$$

$$= \hat{a}\underbrace{\hat{N}|u_\lambda\rangle}_{\lambda|u_\lambda\rangle} - \hat{a}|u_\lambda\rangle =$$

$$\Rightarrow \hat{N}|v_\lambda\rangle = (\lambda-1)\,\hat{a}|u_\lambda\rangle = (\lambda-1)\,|v_\lambda\rangle$$

q.e.d.

__Lemma:__ Let $|w_\lambda\rangle = \hat{a}^+|u_\lambda\rangle$

$$\Rightarrow \hat{N}|w_\lambda\rangle = (\lambda+1)\,|u_\lambda\rangle$$

__Proof:__ $\hat{N}|w_\lambda\rangle = \hat{N}\hat{a}^+|u_\lambda\rangle = (\hat{a}^+\hat{N} + [\hat{N},\hat{a}^+])|u_\lambda\rangle$

$$= (\hat{a}^+\hat{N} + \hat{a}^+)|u_\lambda\rangle$$

$$= (\lambda+1)\,\hat{a}^+|u_\lambda\rangle$$

$$= (\lambda+1)|w_\lambda\rangle$$

q.e.d

Final lemma: The eigenvalues of $\hat{N}$ are non-negative integers

$$\lambda = 0, 1, 2, \ldots, \} = n$$

Proof: Suppose λ were not an integer

Consider $|\Phi\rangle = \hat{a}^m |u_\lambda\rangle$

$$\Rightarrow \hat{N}|\Phi\rangle = (\lambda - m)|u_\lambda\rangle$$

$$\Rightarrow \exists m \text{ such that } \lambda - m < 0$$

contradiction since $\hat{N}$ is positive

If λ = integer = n

$$\Rightarrow \text{When } m = n \qquad \hat{N}\hat{a}^n|u_n\rangle = 0$$

Let $|u_0\rangle = \hat{a}^{n-1}|u_n\rangle$

$$\Rightarrow \hat{N}(\hat{a}|u_0\rangle) = 0$$

However since $\hat{N}$ is positive $\Rightarrow$

$$\Rightarrow \hat{a}|u_0\rangle = 0$$

$\Rightarrow \exists$ a ground state defined by

$\Rightarrow \lambda$ must be an integer

Thus we now know why $\hat{N}$ is referred to as the "number operator"

$$\hat{N}\,|u_n\rangle = n\,|u_n\rangle$$

$$n = 0, 1, 2, 3, \ldots$$

where $\hat{a}\,|u_0\rangle = 0$

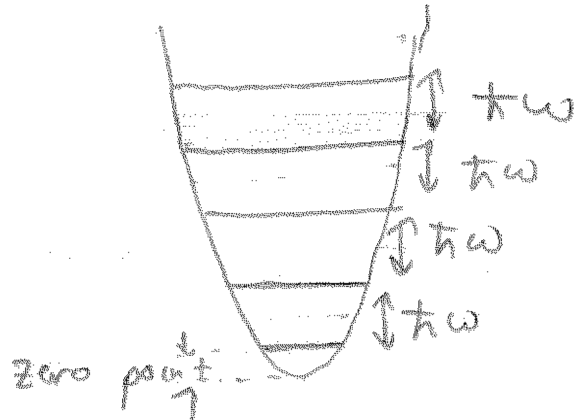
$\Rightarrow \hat{a} \equiv$ "annihilation operator"

<u>Energy eigenvalues</u>

Let $|n\rangle \equiv |u_n\rangle$

$$\hat{H}\,|n\rangle = E_n\,|n\rangle$$

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) = \left(\frac{\hbar\omega}{2}, \frac{3}{2}\hbar\omega, \frac{5}{2}\hbar\omega, \ldots\right)$$



Unique feature of SHO:

<u>Equally spaced</u> energy levels

Eigenvectors: $\left.\begin{array}{l}\text{Raising (creation)}\\ \text{Lowering (annihilation)}\end{array}\right\}$ operators

Consider $\hat{a}|n\rangle = c_n |n-1\rangle$

$$\underbrace{(\langle n|\hat{a}^\dagger)(\hat{a}|n\rangle)} = |c_n|^2 \underbrace{\langle n-1|n-1\rangle}_{=1}$$

$$= \langle n|\hat{a}^\dagger \hat{a}|n\rangle$$

$$= \langle n|\hat{N}|n\rangle = n$$

$\therefore \quad |c_n| = \sqrt{n}$ (take real $\Rightarrow c_n = \sqrt{n}$)

$$\Rightarrow \quad \boxed{\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle}$$

$\boxed{\hat{a} = \text{lowering operator} = \text{annihilation}}$

Similarly $\hat{a}^\dagger|n\rangle = k_n |n+1\rangle$

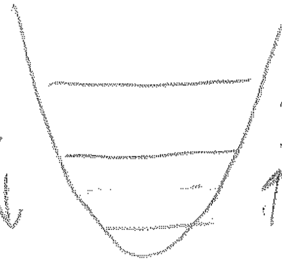
$$(\langle n|\hat{a})(\hat{a}^\dagger|n\rangle) = \langle n|(\hat{a}^\dagger\hat{a} + 1)|n\rangle$$

$$= |k_n|^2 \langle n+1|n+1\rangle = (n+1)\langle n|n\rangle$$

$\Rightarrow \quad |k_n| = \sqrt{n+1}$, chose $k_n = \sqrt{n+1}$

$$\Rightarrow \quad \boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$\boxed{\hat{a}^\dagger = \text{raising operator} = \text{creation}}$

$\hat{a} \downarrow$  $\uparrow \hat{a}^{+}$

$\hat{a} \downarrow$ $\uparrow \hat{a}^{+}$

$(\hat{a}, \hat{a}^{+})$ = "ladder operators"

Finally, given $\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\Rightarrow \hat{a}^{+}|0\rangle = |1\rangle$$

$$(\hat{a}^{+})^{2}|0\rangle = \hat{a}^{+}|1\rangle = \sqrt{2}\,|2\rangle$$

$$(\hat{a}^{+})^{3}|0\rangle = \hat{a}^{+}\sqrt{2}\,|2\rangle = \sqrt{3\cdot 2}\,|3\rangle$$

$$\Rightarrow (\hat{a})^{+n}|0\rangle = \sqrt{n!}\,|n\rangle$$

$\Rightarrow$ We can express the n^{th} level in terms of the ground state

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0\rangle}$$