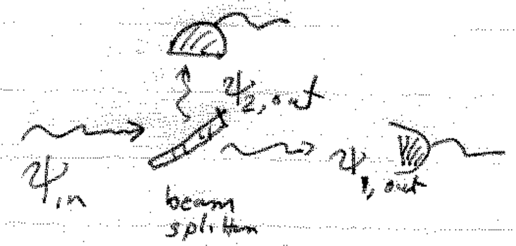
Physics 491: Quantum Mechanics I

Problem Set #1 : Solutions

Problem 1: Mach-Zender Interferometer

(a)



According to the rules given

$$\psi_{1,out} = \frac{1}{\sqrt{2}}\psi_{in} \qquad \psi_{2out} = \frac{1}{\sqrt{2}}\psi_{in}$$

since nothing enters port #2

By the "Born rule", the probability to find a photon at position 1 or 2

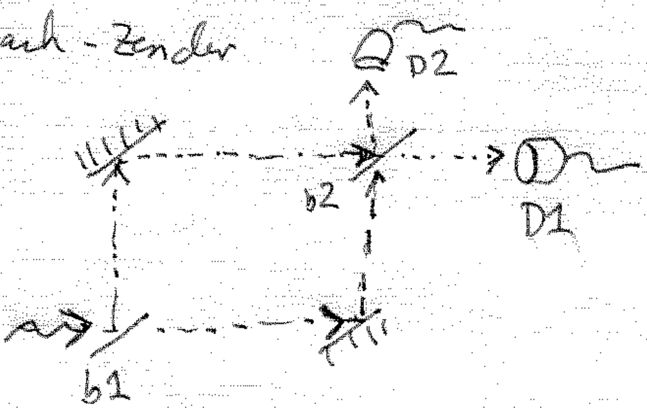
$$P_{1,out} = \int |\psi_{1,out}|^2 dx = \frac{1}{2}\int dx\, |\psi_{in}|^2 = \frac{1}{2}$$

$$P_{2,out} = \int |\psi_{2out}|^2 dx = \frac{1}{2}\int dx\, |\psi_{in}|^2 = \frac{1}{2}$$

$\Rightarrow$ 50-50% chance

Note: The photon is found at one detector or the other, never both. The photon is indivisible. This contrasts classical waves where half of the intensity goes one way and half the other; an antenna would also receive energy. We interpret this as the mean value of a large number of photons.

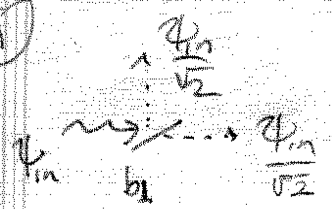
(b) Mach-Zender



The wave function is split at b1, sent along two different paths, and recombined at b2

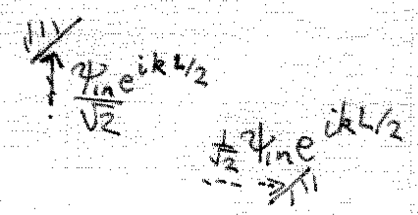
To find the wave functions impinging on D1 and D2 lets apply the transformation rules sequentially.

(1)



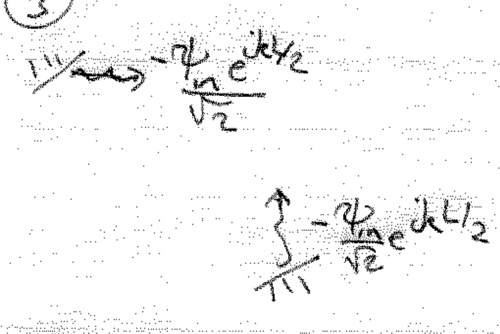
beam-splitter # 1

(2)



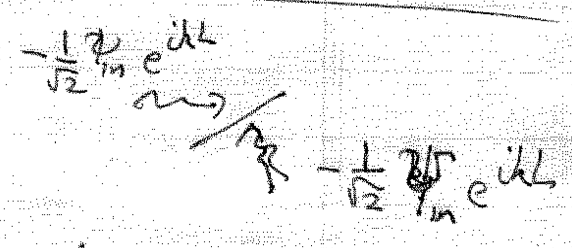
Propagation a distance $L/2$
$\Rightarrow$ phase $e^{ikL/2}$

(3)



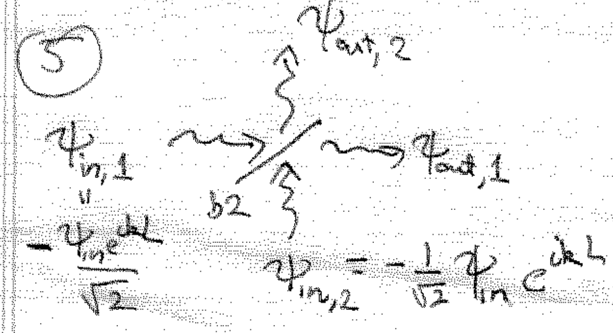
Bounce off mirrors

(4)



Another propagation by a distance $L/2$
$\Rightarrow$ phase $e^{ikL/2}$

(5)



$$\psi_{out,1} = \frac{\psi_{in,1} + \psi_{in,2}}{\sqrt{2}} = e^{ikL} \psi_{in}$$

$$\psi_{out,2} = \frac{\psi_{in,1} - \psi_{in,2}}{\sqrt{2}} = 0$$

$$\Rightarrow P_{out,1} = \int dx\, |\psi_{out,1}|^2 = \int dx\, |\psi_{in}|^2 = \underline{1}$$

$$P_{out,2} = \int dx\, |\psi_{out,2}|^2 = 0$$

$\Rightarrow$

100% chance of detector D1 firing
0% " " " D2 "
No <u>randomness</u>

(c) Classical reasoning

$$P_{D1} = \underbrace{P(\text{transmission at } b2 \mid \text{green})}_{\text{(prob of trans. at b2 given green path)}} \times \underbrace{P(\text{green})}_{\text{(Prob green path was taken)}}$$

$$+ \underbrace{P(\text{reflection at } b2 \mid \text{blue})}_{\text{(Prob. of refl. at b2 given blue path)}} \times \underbrace{P(\text{blue})}_{\text{(Prob blue path was taken)}}$$

Now we ~~know~~ know that there is a 50-50 probability for the photon to take the blue or green path

$$\Rightarrow P(\text{blue}) = P(\text{green}) = \frac{1}{2}$$

Also with the particle incident at b2 along the ~~blue~~ green path there is a 50% chance of transmission and similarly for reflection of blue path

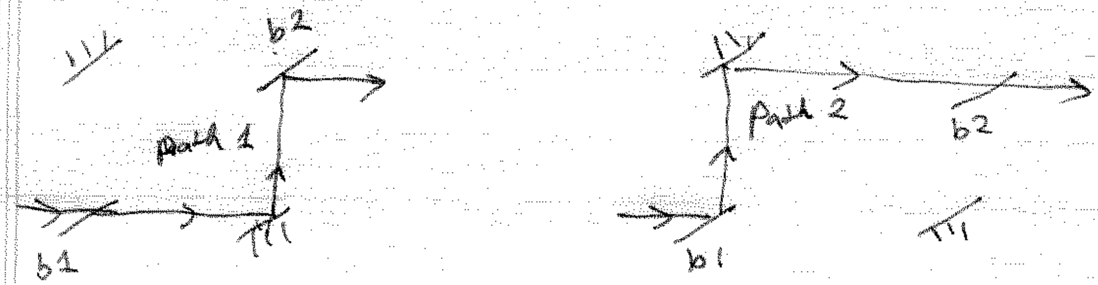
$$\Rightarrow P(\text{trans } b2 \mid \text{green}) = P(\text{refl } b2 \mid \text{blue}) = \frac{1}{2}$$

$$\therefore \quad P_{D1} = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$ Classical reasoning $\Rightarrow$ 50-50 chance of D1 firing

<u>Completely Random!</u>

The quantum case is different because the two paths which lead to detector D1 <u>interfere</u>



$$\psi_{total} = \frac{\frac{1}{\sqrt{2}}\psi_{in}e^{ikL} + \frac{1}{\sqrt{2}}\psi_{in}e^{ikL}}{\sqrt{2}}$$

$$P_{D1} = \int dx\,|\psi_{total}|^2 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \underbrace{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}}_{\text{interference terms}}$$

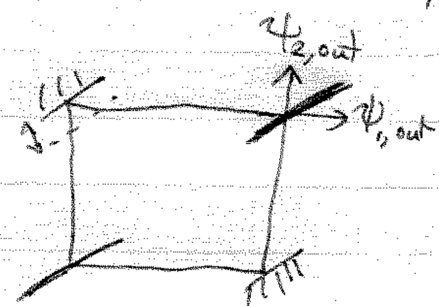
$$\Rightarrow P_{D1} = 1$$

the paths that lead to detector D2 <u>destructively</u> interfere. $\Rightarrow P_{D2} = 0$

(d) Extra credit:

We now want <u>constructive</u> interference for the paths that lead to D2 and <u>destructive</u> for D1.

We can achieve this by changing the <u>relative phase</u> of the two paths by moving the mirror so that the path lengths are <u>unbalanced.</u>



$$\psi_{1,out} = \frac{\frac{1}{\sqrt{2}}\psi_{in} e^{ik(L+\Delta L)} + \frac{1}{\sqrt{2}}\psi_{in} e^{ikL}}{\sqrt{2}}$$

$$\psi_{2,out} = \frac{\frac{1}{\sqrt{2}}\psi_{in} e^{ik(L+\Delta L)} - \frac{1}{\sqrt{2}}\psi_{in} e^{ikL}}{\sqrt{2}}$$

$$P_{D1} = \int dx\, |\psi_{2,out}|^2 = \frac{1}{4}\int dx\, |\psi_{in} e^{ik(L+\Delta L)} + \psi_{in} e^{ikL}|^2$$

$$= \frac{1}{4}\underbrace{\int dx\, |\psi_{in}|^2}_{1}\ \underbrace{|e^{ikL}|^2}_{1}\ |e^{ik\Delta L} + 1|^2$$

$$= \frac{1}{4}(1 + e^{ik\Delta L})(1 + e^{ik\Delta L})^*$$

$$= \frac{1}{4}(1 + e^{ik\Delta L})(1 + e^{-ik\Delta L})$$

$$= \frac{1}{4}(1 + 1 + e^{ik\Delta L} + e^{-i\Delta kL})$$

$$P_{D1} = \frac{1 + \cos(k\Delta L)}{2}$$

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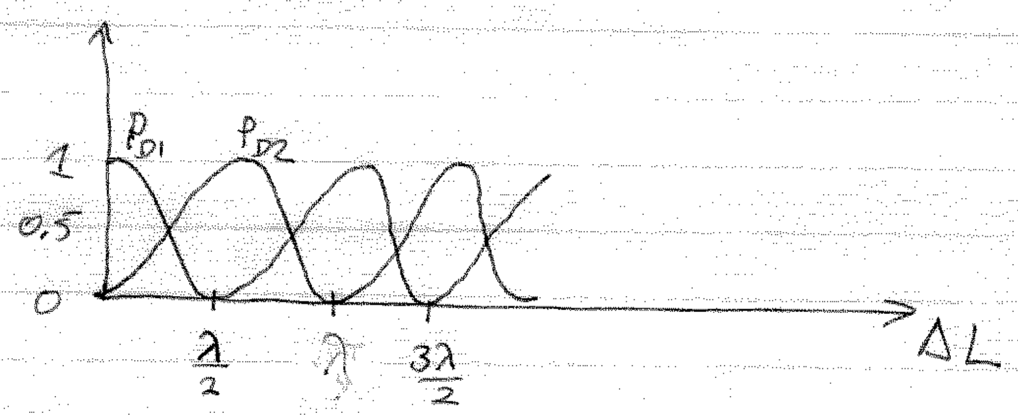
$$= \cos^2\left(\frac{k\Delta L}{2}\right)$$

Similarly $P_{D2} = \frac{1-\cos(k\Delta L)}{2} = \sin^2\left(\frac{k\Delta L}{2}\right)$

Thus, the achieve $P_{D1} = 0$ $P_{D2} = 1$

$\boxed{\begin{array}{l} \text{choose } k\Delta L = m\pi \quad (m \text{ odd}) \\ \Rightarrow \Delta L = m\frac{\lambda}{2} \end{array}}$

Generally the probability of detection in D1 + D2 as a function of ΔL

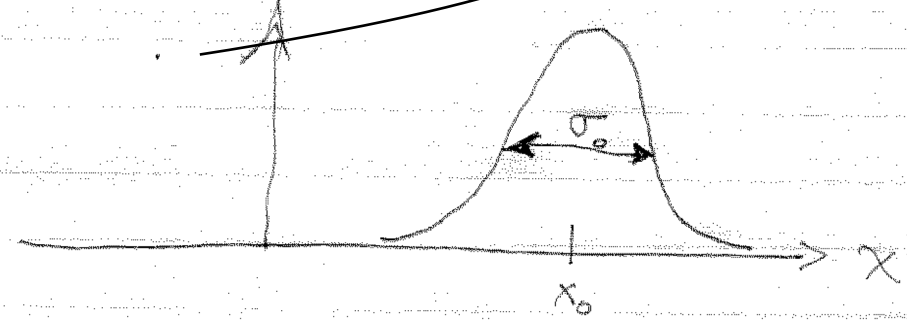


These are the "interference fringes" associated with this interferometer

Problem 2: Bayes' rule with Gaussians

We are trying to determine the position of a particle along one dimension. Our "prior" probability distribution is a Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-x_0)^2}{2\sigma_0^2}}$$



We now measure the position and find ~~the~~ in my detector the value y. However, my detector has only finite resolution, so when my detector reads "y" the true position "x" may still be different. Given an uncertainty Δx in my detector with a Gaussian dist.

Let the "likelihood" distribution be

$$p(y|x) = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(y-x)^2}{2\Delta^2}}$$

Thus, according to Bayes' rule, the updated probability assignment is

$$p(x|y) = N\, p(y|x)\, p(x)$$

where $N^{-1} = \int dx\; p(y|x)\, p(x)$ (normalization)

Let us first calculate

$$p(y|x)\,p(x) = \frac{1}{2\pi\sigma_0\Delta} \exp\left\{ -\frac{(x-x_0)^2}{2\sigma_0^2} - \frac{(y-x)^2}{2\Delta^2} \right\}$$

Aside: $$\frac{(x-x_0)^2}{\sigma_0^2} + \frac{(y-x)^2}{\Delta^2} = \frac{x^2 - 2xx_0 + x_0^2}{\sigma_0^2} + \frac{y^2 - 2yx + x^2}{\Delta^2}$$

$$= \frac{x^2}{\sigma'^2} - 2x\left(\frac{x_0}{\sigma_0^2} + \frac{y}{\Delta^2}\right) + \frac{x_0^2}{\sigma_0^2} + \frac{y^2}{\Delta^2}$$

$$= \frac{x^2}{\sigma'^2} - 2x A(y) + B(y)$$

where $\frac{1}{\sigma'^2} = \frac{1}{\sigma_0^2} + \frac{1}{\Delta^2}$ $\quad A(y) = \frac{x_0}{\sigma_0^2} + \frac{y}{\Delta^2}$

$$B(y) = \frac{x_0^2}{\sigma_0^2} + \frac{y^2}{\Delta^2}$$

Trick: "Complete the square"

$$\frac{x^2}{\sigma'^2} - 2x A(y) = \frac{(x - x')^2}{\sigma'^2} - \frac{x'^2}{\sigma'^2}$$

where $x' = \sigma'^2 A(y) = \left(\frac{x_0}{\sigma_0^2} + \frac{y}{\Delta^2}\right)\sigma'^2$

$$\Rightarrow x' = x_0 + x_0\left(1 - \frac{\sigma'^2}{\sigma_0^2}\right) + y\,\frac{\sigma'^2}{\Delta^2}$$

Now $\sigma'^2 = \frac{\sigma_0^2 \Delta^2}{\sigma_0^2 + \Delta^2}$

$$\Rightarrow x' = x_0 + K_1(y - x_0)$$

where $K_1 = \sigma_0^2 / (\Delta^2 + \sigma_0^2$

Putting this all together:

$$p(y|x)\,p(x) = \underset{\Uparrow}{N}(y, x_0) \quad e^{-\frac{(x-x')^2}{2\sigma'^2}}$$

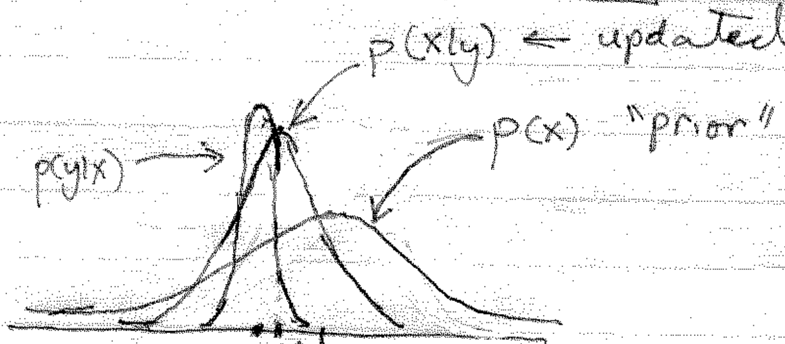
All the rest of the factors

Instead of keeping track of all these factors, we need just replace it by the normalization

$$\Rightarrow p(x|y) = \frac{1}{\sqrt{2\pi\sigma'^2}} \quad e^{-\frac{(x-x')^2}{2\sigma'^2}}$$

$$\boxed{\begin{aligned} \text{where} \quad x' &= x_0 + K_1(y - x_0) \\ K &= \frac{\sigma_0^2}{\Delta^2 + \sigma_0^2} \\ \sigma'^2 &= \frac{\sigma_0^2 \Delta^2}{\sigma_0^2 + \Delta^2} = K_2 \sigma_0^2 \end{aligned}}$$

Graphically:



After the measurement, the new distribution is a narrower Gaussian peaked closer to the actual position.