Physics 491: Quantum Mechanics I

Problem Set #2
Due: Wednesday, September 12, 2018

Problem 1: Properties of the Fourier Transform (20 points)

Given a "well behaved" function $F(x)$, we defined the Fourier transform $\tilde{F}(k) = \int_{-\infty}^{\infty} dx\, F(x) \frac{e^{-ikx}}{\sqrt{2\pi}}$ and inverse $F(x) = \int_{-\infty}^{\infty} dk\, \tilde{F}(k) \frac{e^{ikx}}{\sqrt{2\pi}}$. Prove the following properties.

(a) Shift-phase duality:
If $F(x) = f(x - x_0)$ then $\tilde{F}(k) = \tilde{f}(k)e^{-ikx_0}$. If $F(x) = f(x)e^{ik_0 x}$ then $\tilde{F}(k) = \tilde{f}(k - k_0)$.

(b) Convolution:
If $F(x) = f(x)g(x)$ then $\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')\tilde{g}(k - k')$. Apply this to the case where $g(x) = e^{ik_0 x}$ to reconfirm the shift-phase property.

(c) Derivatives: If $F(x) = \frac{d^n f(x)}{dx^n}$, then $\tilde{F}(k) = (ik)^n \tilde{f}(k)$.

(d) Parseval's Theorem.
$\int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{F}(k)|^2 dk$. Use this to show that when $F(x) = \psi(x)$ is a normalized wave function, the momentum-space wave function is also normalized.

Problem 2: Square wave packet (20 points)

Consider a free particle. initially with a well defined momentum p_0, whose wave function is well approximated by a plane wave. At t=0, the particle is the *localized* in a region $-\frac{a}{2} \le x \le \frac{a}{2}$, so that is wave function is

$$\psi(x) = \begin{cases} Ae^{ip_0 x/\hbar}, & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the normalization constant A and sketch $\mathrm{Re}(\psi(x))$, $\mathrm{Im}(\psi(x))$, and $|\psi(x)|^2$.
(b) Find the momentum space wave function $\tilde{\phi}(p)$ and show that it too is normalized.
(c) *Estimate* the uncertainties Δx and Δp at this time. How close is this to a minimum uncertainty wave packet? If you calculate the standard deviations exactly you will find $\Delta p \to \infty$. Why do you think this is the case?
(d) Estimate the spreading of the wave packet's rms as a function of time, $\Delta x(t)$.

Problem 3: The Dirac delta function. (15 points)

The Dirac delta function is defined by, $\int_{-\infty}^{\infty} dx\, f(x)\delta(x-x_0) = f(x_0)$. Actually, $\delta(x)$ is not a well behaved function at all, but only make sense in the integral. It can be though of as the limit of a true function $\delta(x) = \lim_{\varepsilon \to 0} \delta_e(x)$. In class we chose,

$$\delta_e(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\, e^{-\varepsilon|k|} = \frac{\varepsilon/\pi}{x^2+\varepsilon^2}.$$

This is not the only possibility that limits to the delta function.

Show that the following other choices limit to a delta function as $\varepsilon \to 0$. Sketch them. Find also their Fourier transforms and comment and their behavior as $\varepsilon \to 0$.

(i) $\delta_\varepsilon(x) = \frac{1}{\varepsilon}, -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2}, \quad 0 \text{ otherwise}.$

(ii) $\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2}.$

(iii) $\delta_\varepsilon(x) = \frac{\sin(x/\varepsilon)}{\pi x}.$

(iv) **Extra credit (5 points):** $\delta_\varepsilon(x) = \frac{\varepsilon^2 \sin^2(x/\varepsilon)}{\pi x^2}$