Physics 491

Problem Set #2 Solutions

Problem 1: Properties of the Fourier Transform

$$\tilde{F}(k) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} F(x)\, e^{-ikx}$$

(a) Shift-phase duality

- Let $F(x) = f(x-x_0) \Rightarrow \tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x-x_0)\, e^{-ikx}$

let $y = x - x_0 \Rightarrow \tilde{F}(k) = \underbrace{\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f(y)\, e^{-iky}}\, e^{-ikx_0} = \tilde{f}(k)\, e^{-ikx_0}$ ✓

- Let $F(x) = f(x)e^{ik_0x} \Rightarrow \tilde{F}(k) = \underbrace{\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x)\, e^{-i(k-k_0)x}}_{\tilde{f}(k-k_0)}$ ✓

(b) Convolution: $F(x) = f(x)g(x)$ (product of two functions)

$$\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x)g(x)\, e^{-ikx}$$

Now plug in

$$\begin{cases} f(x) = \int \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k')\, e^{ik'x} \\ g(x) = \int \frac{dk''}{\sqrt{2\pi}} \tilde{g}(k'')\, e^{ik''x} \end{cases}$$

$$\Rightarrow \tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dx\, dk'\, dk''}{(2\pi)^{3/2}} \tilde{f}(k')\, \tilde{g}(k'')\, e^{-i(k-k'-k'')x}$$

$$= \int \frac{dk'\, dk''}{(2\pi)^{3/2}} \tilde{f}(k')\, \tilde{g}(k'') \underbrace{\int_{-\infty}^{\infty} dx\, e^{-i(k-k'-k'')x}}_{2\pi\,\delta(k-k'-k'')}$$

$$\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k') \int_{-\infty}^{\infty} dk'' \tilde{g}(k'') \delta(k-k'-k'')$$

$$\Rightarrow \boxed{\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk'}{\sqrt{2\pi}} \tilde{f}(k') \tilde{g}(k-k')}$$

This integral is known as a <u>convolution</u>. It constitute shifting one function relative to the other and then finding the area under the curve.

Application: Let $g(x) = e^{ik_0 x}$ $\Rightarrow$ $\tilde{g}(k) = \sqrt{2\pi}\,\delta(k-k_0)$

$$\Rightarrow \tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk'}{\cancel{\sqrt{2\pi}}} \tilde{f}(k') \cancel{\sqrt{2\pi}}\,\delta(k-k'-k_0) = \tilde{f}(k-k_0)$$

(c) $F(x) = \frac{d^n f(x)}{dx^n} \Rightarrow \tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{d^n}{dx^n} f(x)\, e^{-ikx}$

Consider $n=1$: $\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{df}{dx} e^{-ikx}$

Integration by parts: $\frac{df}{dx} e^{-ikx} = -f(x)\frac{d}{dx}\left(e^{-ikx}\right) + \underbrace{\frac{d}{dx}\left[f(x)e^{-ikx}\right]}_{\Downarrow}$

$$= ik\, f(x)\, e^{-ikx} + $$

$$\Rightarrow \tilde{F}(k) = ik \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} f(x)\, e^{-ikx} + \underbrace{\left[f(x) e^{-ikx}\right]_{-\infty}^{+\infty}}_{0}$$

$$= ik\,\tilde{f}(k)$$

assuming $f(x)$ normalizable

Repeating n times

$$\Rightarrow \boxed{\tilde{F}(k) = (ik)^n \tilde{f}(k)}$$

The fact that derivatives turn into algebraic expressions makes the Fourier transform very useful for solving differential equations

(d) Parseval's theorem

Consider: $\int_{-\infty}^{\infty} |F(x)|^2 \, dx = \int_{-\infty}^{\infty} F(x) \, F^*(x) \, dx$

Plug in $F(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{F}(k) \, e^{ikx}$

$$= \int_{-\infty}^{\infty} \frac{dx \, dk \, dk'}{2\pi} \tilde{F}(k) \, \tilde{F}^*(k') \, e^{i(k-k')x}$$

$$= \int_{-\infty}^{\infty} dk \, dk' \, \tilde{F}(k) \, \tilde{F}^*(k') \underbrace{\int_{-\infty}^{\infty} \frac{dx}{2\pi} \, e^{i(k-k')x}}_{\delta(k-k')}$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{F}(k)|^2 dk$$

Consider the situation when $F(x) = \Psi(x)$ (a quantum wave function)

The momentum space wave function

$$\tilde{\phi}(p) = \frac{1}{\sqrt{\hbar}} \tilde{\Psi}(k = p/\hbar)$$

$$\text{or} \quad \tilde{\Psi}(k) = \sqrt{\hbar} \, \tilde{\phi}(p = \hbar k)$$

$$\therefore \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{\Psi}(k)|^2 dk = \int_{-\infty}^{\infty} |\tilde{\phi}(p = \hbar k)|^2 \hbar \, dk$$

$$= \int_{-\infty}^{\infty} |\tilde{\phi}(p)|^2 dp \quad \checkmark$$

Problem 3: The Dirac delta function

$\delta(x-x_0)$ is defined by $\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-x_0) = f(x_0)$

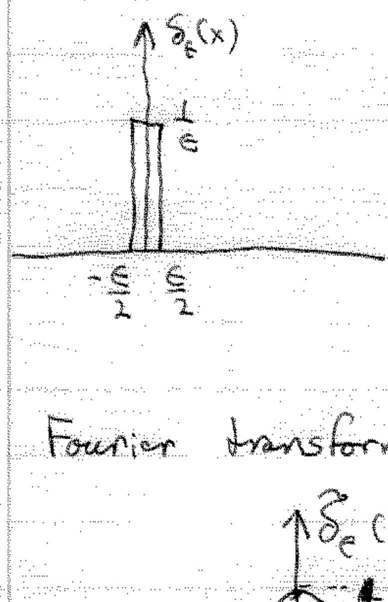
that means $\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$

such that $\int_{-\infty}^{\infty} dx\, \delta(x-x_0) = 1$

We want to consider representations of $\delta(x)$ as the limit $\lim_{\epsilon \to 0} \delta_\epsilon(x)$ where $\delta_\epsilon(x)$ is a true function.

$\delta_\epsilon(x)$ must satisfy
- Peaked at $x = 0$
- Width $\to 0$, Height $\to \infty$ as $\epsilon \to 0$
- Integral under curve $= 1$

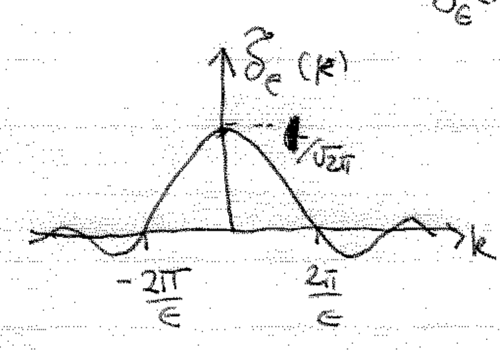
(i) Consider $\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} < x < \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases}$



Clearly $\delta_\epsilon(x)$ peaked at origin and width $\to 0$ as $\epsilon \to 0$ ✓

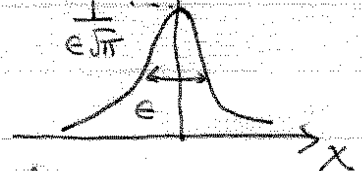
$\int_{-\infty}^{\infty} \delta_\epsilon(x)\, dx = 1$ ✓

Fourier transform: $\tilde{\delta}_\epsilon(k) = \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} dx\, \frac{e^{ikx}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}\left(\frac{k\epsilon}{2}\right)$



As $\epsilon \to 0$, spread in k $\to \infty$ as expected

(ii) $\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-\frac{x^2}{\epsilon}}$ Gaussian



Normalized! $\sigma^2 = \frac{\epsilon}{2}$

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\,dx = 1 \checkmark$$

Peaked at origin

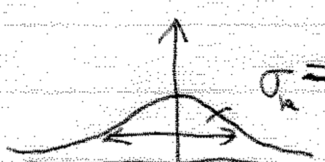
As $\epsilon \to 0$ height $\to \infty$, width $\to 0$ ✓

Fourier transform $\tilde{\delta}_\epsilon(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \frac{e^{-x^2/\epsilon}}{\epsilon\sqrt{\pi}} e^{-ikx}$

Aside $-\frac{x^2}{\epsilon} - ikx = -\frac{1}{\epsilon}\left(x + \frac{i\epsilon}{2}k\right)^2 - \frac{\epsilon k^2}{2}$

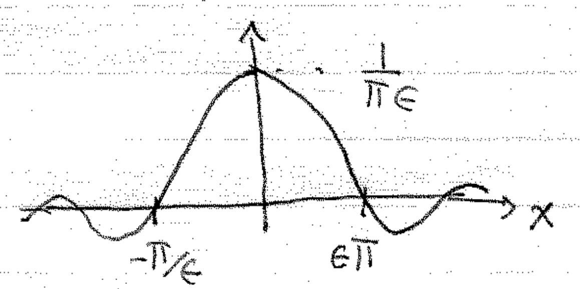
"complete the square"

$$\Rightarrow \tilde{\delta}_\epsilon(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon k^2}{2}} \underbrace{\int_{-\infty}^{\infty} dx \frac{e^{-\frac{1}{\epsilon}\left(x - \frac{i\epsilon k}{2}\right)^2}}{\sqrt{\pi \epsilon^2}}}_{=1 \text{ (normalize Gaussian)}}$$



$\sigma_k = \frac{\sqrt{2}}{\epsilon}$, width in k $\to \infty$

(iii) $\delta_\epsilon(x) = \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} = \frac{1}{\pi\epsilon} \operatorname{sinc}\left(\frac{x}{\epsilon}\right)$



Peak at $x = 0$

Height $\to \infty$, Width $\to 0$ } as $\epsilon \to 0$ ✓

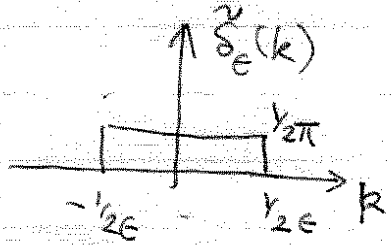
Area $\frac{1}{\pi\epsilon} \int_{-\infty}^{\infty} dx \operatorname{sinc}\left(\frac{x}{\epsilon}\right) = \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} dy \operatorname{sinc}(y)}_{\pi} = 1 \checkmark$

Fourier transform: $\tilde{\delta}_\epsilon(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, \frac{1}{\pi\epsilon} \operatorname{sinc}\left(\frac{x}{\epsilon}\right) e^{-ikx}$

$$= \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \operatorname{sinc}(x) \, e^{-i(\epsilon k) y}}_{\text{Fourier transform of sinc function evaluated at } k \to \epsilon k}$$

Now, the Fourier transform of a sinc function must be a rectangular pulse from Prob. 2

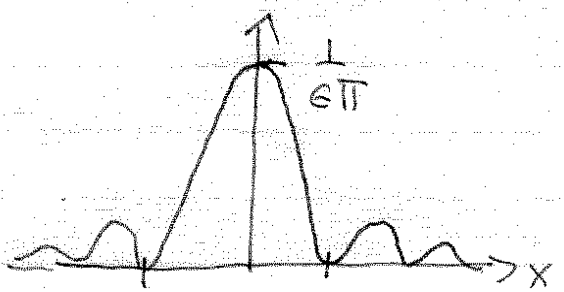
$$\Rightarrow \quad \tilde{\delta}_\epsilon(k) = \begin{cases} \frac{1}{2\pi} & -\frac{1}{2} < \epsilon k < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{2\pi} & -\frac{1}{2\epsilon} \le k \le \frac{1}{2\epsilon} \\ 0 & \text{otherwise} \end{cases}$$



as $\epsilon \to 0$

$\Rightarrow$ Infinite spread in k

(iv) $\delta_\epsilon(x) = \frac{\epsilon}{\pi} \frac{\sin^2\left(\frac{x}{\epsilon}\right)}{x^2} = \frac{1}{\epsilon\pi} \operatorname{sinc}^2\left(\frac{x}{\epsilon}\right)$



Peaked at $x = 0$

$\left.\begin{array}{l}\text{Height} \to \infty \\ \text{Width} \to 0\end{array}\right\}$ at $\epsilon \to 0$

$$\int_{-\infty}^{\infty} dx \; \delta_\epsilon(x) = 1$$

(Next Page)

To find the Fourier transform, let's use the convolution theorem.

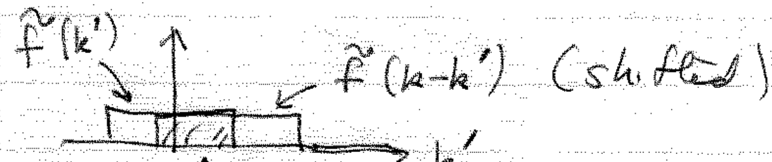
$$\delta_\epsilon(x) = \frac{1}{\epsilon\pi}\,\mathrm{sinc}^2\!\left(\frac{x}{\epsilon}\right) = \frac{1}{\epsilon\pi}\,\mathrm{sinc}\!\left(\frac{x}{\epsilon}\right)\mathrm{sinc}\!\left(\frac{x}{\epsilon}\right)$$

$$\Rightarrow \tilde{\delta}_\epsilon(k) = \frac{1}{\epsilon\pi}\,\tilde{f}_\epsilon(k) \underset{\substack{\uparrow\\ \text{convolution}}}{\circledast} \tilde{f}_\epsilon(k)$$

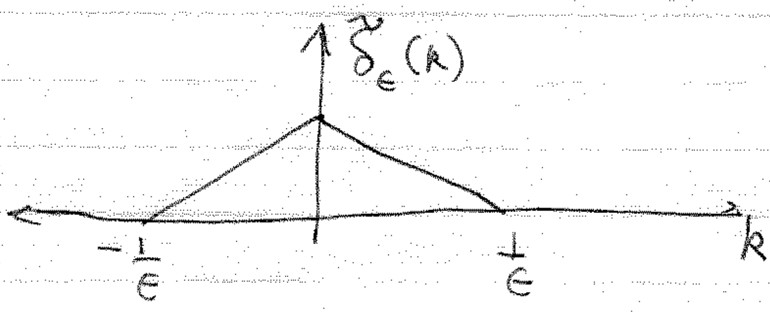
where $\tilde{f}_\epsilon(k)$ is the Fourier transform of $\mathrm{sinc}\left(\frac{x}{\epsilon}\right)$

$$= \epsilon\,(\text{F.T. of } \underbrace{\mathrm{sinc}(x)}_{=\text{ rectangular pulse}})$$

The convolution of two rectangular pulses is a <u>triangular pulse</u>



Area under product of curve and shifted curve



width $\to \infty$ as $\epsilon \to 0$ ✓