Physics 491: Quantum Mechanics 1
Problem Set #3: Solutions

Problem 1 Square packet

Intial plane wave $\psi_0 e^{ip_0 x/\hbar}$ is localized in a region of with a

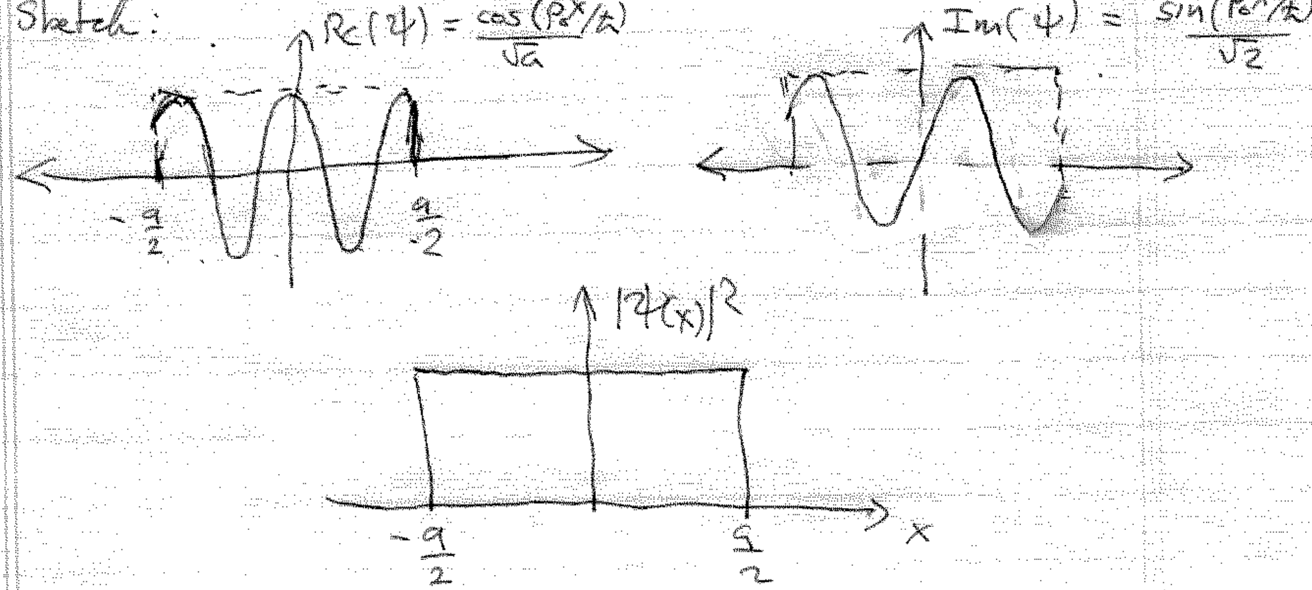
$$\psi(x) = \begin{cases} A e^{ip_0 x/\hbar} & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

(a) Normalization $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-a/2}^{a/2} dx \; |A|^2 = 1$

$\Rightarrow a|A|^2 = 1 \Rightarrow$ $\boxed{A = \frac{e^{i\phi}}{\sqrt{a}}}$ ← overall phase is arbitrary (set $\phi = 0$)

$\Rightarrow \psi(x) = \frac{1}{\sqrt{a}} e^{ip_0 x/\hbar}, \quad -\frac{a}{2} \le x \le \frac{a}{2}$

Sketch:



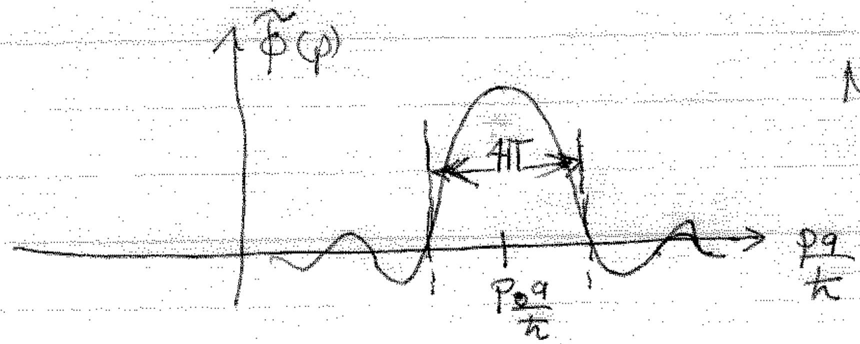
Note that the probability density in "position space" does not depend on the momentum p_0

(b) Momentum space $\quad \tilde{\phi}(p) = \frac{1}{\sqrt{\hbar}}\tilde{\phi}(k = p/\hbar)$

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, \psi(x)\, e^{-ipx/\hbar} = \frac{1}{\sqrt{2\pi\hbar a}}\int_{-a/2}^{a/2} dx\, e^{-i\frac{(p-p_0)x}{\hbar}}$$

$$= \frac{1}{\sqrt{2\pi\hbar a}}\left[\frac{\hbar}{-i(p-p_0)}\, e^{-i\frac{(p-p_0)x}{\hbar}}\right]_{-a/2}^{a/2} = 2\sqrt{\frac{1}{2\pi a\hbar}}\; \frac{\sin\left[\frac{(p-p_0)}{\hbar}\frac{a}{2}\right]}{\frac{(p-p_0)}{\hbar}}$$

$$\Longrightarrow \tilde{\phi}(p) = \sqrt{\frac{a}{2\pi\hbar}}\, \mathrm{sinc}\left[\frac{(p-p_0)a/2}{\hbar}\right] \quad \text{where } \mathrm{sinc}(x) = \frac{\sin x}{x}$$



Note: the phase - shift duality

Normalization

$$\int_{-\infty}^{\infty} dp\, |\tilde{\phi}(p)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a}{\hbar}\, dp\, \mathrm{sinc}^2\left[\frac{(p-p_0)a/2}{\hbar}\right]$$

let $y = \frac{ap}{\hbar}$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} dy\, \mathrm{sinc}^2\left[\frac{y-y_0}{2}\right]}_{2\pi \text{ (Mathematica)}} = 1 \; \checkmark$$

So, as expected, given a normalized $\psi(x)$, with all the factor $\tilde{\phi}(p)$ is normalized.

(c) Uncertainties:

We can estimate Δx and Δp "by eye" looking at the probability distributions $|\psi(x)|^2$ and $|\tilde{\Phi}(p)|^2$

Clearly $\Delta x \sim a$
If we take Δp to be the width shown on the previous page $\Rightarrow \Delta p \sim 4\pi \frac{\hbar}{a}$

$\Rightarrow \Delta x \, \Delta p \sim 4\pi \hbar$, Not quite minimum uncertainty

Formally: $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$\langle x\rangle = 0$ (by inspection)

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx \; x^2 |\psi(x)|^2 = \int_{-a/2}^{a/2} dx \; \frac{x^2}{a} = \frac{x^3}{3a}\Big|_{-a/2}^{a/2}$$

$$= \frac{a^2}{12} \quad \Rightarrow \quad \Delta x = \frac{a}{\sqrt{12}}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle}$$

$\langle p\rangle = p_0$ (by inspection)

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dp \; p^2 |\tilde{\phi}(p)|^2 = \int_{-\infty}^{\infty} \frac{dp}{\pi} \frac{\sin^2[(p-p_0)\frac{a}{2}]}{(p-p_0)^2} p^2$$

$$\Rightarrow \Delta p^2 = \langle (p-p_0)^2\rangle = \frac{\hbar}{\pi a}\int_{-\infty}^{\infty} dq \; q^2 \sin^2\left(\frac{qa}{2}\right) = \infty!$$

Thus the formal variance diverges

This is because of the discontinuity: Not physical

(d) Estimate spreading:

Because the initial packet has a distribution of momenta, after some time it will spread apart. (Think about a pack of people running. After some time the fast runners will move ahead and the slow runners will fall behind, so the distribution of people will spread out).

If we follow the trajectory of a point

$$x(t) = \underset{\substack{\uparrow \\ \text{initial}}}{x(0)} + \frac{p(0)}{m} t$$

$\Rightarrow$ Spread: $\Delta x(t) = \sqrt{\Delta x^2(t)}$ $\quad \Delta x^2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$

$$\Rightarrow \Delta x^2(t) = \langle x^2(0) \rangle + \frac{\langle p^2(0) \rangle}{m} t^2 + \frac{2t}{m} \langle x(0)\, p(0) \rangle$$
$$- \langle x(0) \rangle^2 - \frac{\langle p(0) \rangle^2}{m^2} t^2 - \frac{2t}{m} \langle x(0) \rangle \langle p(0) \rangle$$

$$\Rightarrow \Delta x^2(t) = \Delta x^2(0) + \frac{\Delta p^2(0)\, t^2}{m^2} - \frac{2t}{m} \sigma_{xp}(0)$$

where $\sigma_{xp}(0) = \langle x(0)\, p(0) \rangle - \langle x(0) \rangle \langle p(0) \rangle$ = Covariance

~~Assuming~~ In the initial packet x and p are <u>uncorrelated</u>

$$\Longrightarrow \sigma_{xp}(0) = 0$$

$$\therefore \quad \Delta x^2(t) = \Delta x^2(0) + \Delta x^2_{spread}(t)$$

$$\text{where} \quad \Delta x^2_{spread}(t) = \left(\frac{\Delta p(0)}{m} t\right)^2$$

For our problem $\quad \left.\begin{aligned} \Delta x(0) &\sim a \\ \Delta p(0) &\sim \frac{\hbar}{a} \end{aligned}\right\}$ "Estimate"

$$\Rightarrow \Delta x^2(t) \cong a^2 + \left(\frac{\hbar t}{ma}\right)^2$$

$$\Rightarrow \boxed{\Delta x(t) \cong a\sqrt{1 + \frac{\hbar}{ma^4} t^2}}$$

<u>Note</u>: The more localized the packet is initially, the faster it spreads apart since $\Delta p(0)$ is larger.

<u>Note</u>: The spreading of the packet is due to <u>group velocity dispersion</u>

Recall: For free particle $\quad \omega = \frac{\hbar k^2}{2m}$

$$\Rightarrow v_g = \frac{\partial \omega}{\partial k} = \frac{\hbar k}{m} = \frac{p}{m}$$

Spread in group velocities $\quad \Delta v_g = \frac{\partial v_g}{\partial k}\Delta k = \frac{\partial^2 \omega_k}{\partial k^2}\Delta k$

$$= \frac{\hbar}{m}\Delta k = \frac{\Delta p}{m}$$

Spreading $\quad \Delta x_{spread}(t) = \Delta v_g t = \frac{\Delta p(0)}{m} t$

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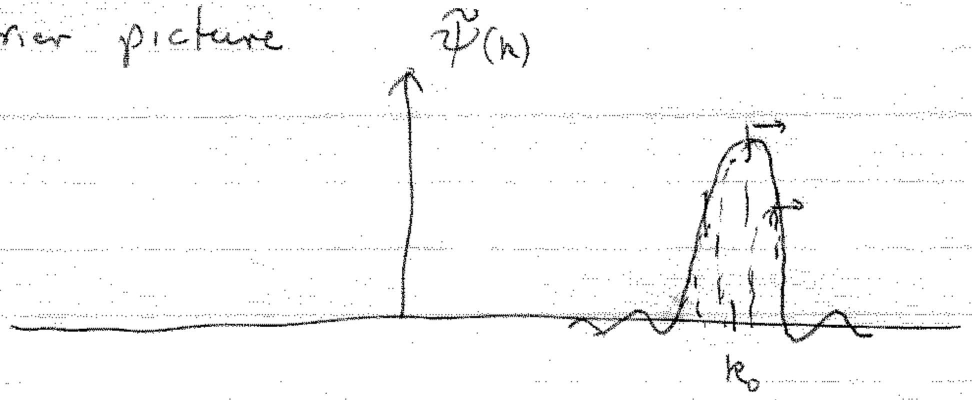
Fourier picture



Inside the distribution there are "subpackets" shown ~~schematically~~ as dashed lines above. Each subpacket moves at its own group velocity $v_g(k)$. Because v_g varies as a function of k, ~~each~~ the packets will spread apart. The spread in group velocities

$$\Delta v_g = \frac{\partial v_g}{\partial k} \Delta k = \frac{\partial^2 \omega}{\partial k^2} \Delta k \qquad (\text{G.V.D.})$$

2018 Fall Physics 491: Problem Set 3

Problem 2

Contents

1 More examples in momentum space

We consider the wave function $\psi(x) = Ax\exp\left(-\frac{x^2}{4\sigma^2}\right)$.

It is useful to recall the form of a normalized Gaussian probability distribution and its properties. The distribution with mean x_0 and variance σ^2 is as follows.

$$\mathcal{P}_{\text{Gauss}} = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right) \tag{1}$$

The moments of this distribution are as follows. The zeroth moment gives the normalization.

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^0 \mathcal{P}_{\text{Gauss}} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathcal{P}_{\text{Gauss}} = 1 \tag{2}$$

The first moment gives the mean.

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^1 \mathcal{P}_{\text{Gauss}} = \int_{-\infty}^{+\infty} \mathrm{d}x\, x\mathcal{P}_{\text{Gauss}} = x_0 \tag{3}$$

The second moment gives the variance.

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \mathcal{P}_{\text{Gauss}} = \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2\mathcal{P}_{\text{Gauss}} = \sigma^2 + x_0^2 \tag{4}$$

1.1 Normalization constant

We find the normalization by requiring that the integral of the square of the absolute value of the wave function integrated over all space is 1.

$$\int_{-\infty}^{+\infty} \mathrm{d}x|\psi(x)|^2 = 1 \implies |A|^2 \int_{-\infty}^{+\infty} \mathrm{d}x\left|x\exp\left(-\frac{x^2}{4\sigma^2}\right)\right|^2 = 1 \tag{5}$$

For our wave function we have the following.

$$\int_{-\infty}^{+\infty} \mathrm{d}x|\psi(x)|^2 = |A|^2 \int_{-\infty}^{+\infty} \mathrm{d}x\left|x\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\right|^2 = |A|^2\sqrt{2\pi\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} \mathrm{d}x x^2\exp\left(-\tfrac{x^2}{4\sigma^2}\right) = |A|^2\sqrt{2\pi\sigma^2}\sigma^2 \tag{6}$$

Therefore the condition for normalization give the following expression for $|A|^2$.

$$|A|^2\sqrt{2\pi\sigma^2}\sigma^2 = 1 \implies |A|^2 = \frac{1}{(2\pi\sigma^6)^{1/2}} \implies |A| = \frac{1}{(2\pi)^{1/4}\sigma^{3/2}} \tag{7}$$

We choosing the overall phase to be zero, that is, choose A to be real and positive.

$$A = \frac{1}{(2\pi)^{1/4}\sigma^{3/2}} \tag{8}$$

Therefore the normalized wavefunction is

$$\psi(x) = \frac{1}{(2\pi)^{1/4}\sigma^{3/2}} x \exp\left(-\tfrac{x^2}{4\sigma^2}\right) \tag{9}$$

It is useful to factor out a Gaussian wavefunction.

$$\psi(x) = \frac{1}{(2\pi)^{1/4}\sigma^{3/2}} x \exp\left(-\tfrac{x^2}{4\sigma^2}\right) = \left(\frac{x}{\sigma}\right)\frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\tfrac{x^2}{4\sigma^2}\right) = \left(\frac{x}{\sigma}\right)\psi_{\text{Gauss}}(x) \tag{10}$$

1.2 Momentum space wave function

We find the momentum space wave function $\tilde{\phi}(p)$ by doing a Fourier transform from position space to momentum space.

$$\tilde{\phi}(p) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}}\psi(x)\exp\left(-i\tfrac{px}{\hbar}\right) = \frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}x\, x\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\exp\left(-i\tfrac{px}{\hbar}\right) \tag{11}$$

To do this integral, we use the following trick.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\right) = -\frac{x^2}{2\sigma^2}\exp\left(-\tfrac{x^2}{4\sigma^2}\right) \tag{12}$$

Therefore the momentum space wavefunction is as follows.

$$\tilde{\phi}(p) = -A\frac{2\sigma^2}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\right)\exp\left(-i\tfrac{px}{\hbar}\right) \tag{13}$$

We use integration by parts and use the fact that the Gaussian function goes to zero as $|x| \to \infty$.

$$\begin{aligned}
\tilde{\phi}(p) = -A\frac{2\sigma^2}{\sqrt{2\pi\hbar}}\left[\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\exp\left(-i\tfrac{px}{\hbar}\right)\right]_{-\infty}^{+\infty} + A\frac{2\sigma^2}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\mathrm{d}x\,\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-i\tfrac{px}{\hbar}\right)\right) \\
= 0 + A\frac{2\sigma^2}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\mathrm{d}x\,\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\left(-\frac{ip}{\hbar}\right)\exp\left(-i\tfrac{px}{\hbar}\right) \\
= 2A\sigma^2\left(-\frac{ip}{\hbar}\right)\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\mathrm{d}x\,\exp\left(-\tfrac{x^2}{4\sigma^2}\right)\exp\left(-i\tfrac{px}{\hbar}\right)}_{\text{Momentum space wave function of a Gaussian}} = 2A\sigma^2\left(-\frac{ip}{\hbar}\right)\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right)
\end{aligned} \tag{14}$$

Putting the factors together, we have the following.

$$\tilde{\phi}(p) = -i\left(\frac{2p\sigma}{\hbar}\right)\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right) \tag{15}$$

Again, it is useful to factor out a Gaussian wavefunction.

$$\tilde{\phi}(p) = -i\left(\frac{2p\sigma}{\hbar}\right)\tilde{\phi}_{\text{Gauss}}(p) \tag{16}$$

Up to overall constant factors, the momentum space wavefunction has the same form as the position space wave function.

1.3 Position and momentum uncertainties

The uncertainties in position and momentum can be found using the following relationships.

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \tag{17}$$

For evaluating these expressions, the following property of Gaussian distribution is useful to note.

$$\langle x - x_0 \rangle^n_{\text{Gauss}} = \begin{cases} 0 & n \text{ is odd} \\ (n-1)!!\sigma^n & n \text{ is even} \end{cases} \tag{18}$$

Here $(n-1)!!$ is the double factorial, $(n-1)!! = (n-1) \times (n-3) \times 3 \times 1$.
In particular the second moment $\langle x \rangle^2$ for a Gaussian distribution with mean x_0 and variance σ^2 is $\sigma^2 + x_0^2$
We note that $\langle x \rangle = 0$ as $|\psi(x)|^2$ is even about $x = 0$. Moreover $\langle p \rangle = 0$ as $|\tilde{\phi}(p)|^2$ is even about $p = 0$

$$\langle x \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x \, x |\psi(x)|^2 = 0 \qquad \langle p \rangle = \int_{-\infty}^{+\infty} \mathrm{d}p \, p |\tilde{\phi}(p)|^2 = 0 \tag{19}$$

The second moment for x is the following.

$$\langle x \rangle^2 = \int_{-\infty}^{+\infty} \mathrm{d}x \, x^2 |\psi(x)|^2 = \int_{-\infty}^{+\infty} \mathrm{d}x \, x^2 \left| \left(\frac{x}{\sigma} \right) \psi_{\text{Gauss}}(x) \right|^2 = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \mathrm{d}x \, x^4 |\psi_{\text{Gauss}}|^2 \tag{20}$$

Using the property of Gaussian distribution, we find the following.

$$\langle x \rangle^2 = \frac{1}{\sigma^2} (4-1)!!\sigma^4 = 3\sigma^2 \tag{21}$$

Therefore, the uncertainty in x, Δx is as follows.

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{3}\sigma \tag{22}$$

The second moment for p is the following.

$$\langle p \rangle^2 = \int_{-\infty}^{+\infty} \mathrm{d}p \, p^2 \left| \tilde{\phi}(p) \right|^2 = \int_{-\infty}^{+\infty} \mathrm{d}p \, p^2 \left| \left(-i \frac{2p\sigma}{\hbar} \right) \tilde{\phi}_{\text{Gauss}}(p) \right|^2 = \frac{4\sigma^2}{\hbar^2} \int_{-\infty}^{+\infty} \mathrm{d}x \, p^4 \left| \tilde{\phi}_{\text{Gauss}} \right|^2 \tag{23}$$

Again, we use the property of the Gaussian distribution. Recall that the variance of the momentum space Gaussian is $\frac{\hbar}{2\sigma}$.

$$\langle p \rangle^2 = \frac{4\sigma^2}{\hbar^2} (4-1)!! \left(\frac{\hbar}{2\sigma} \right)^4 = \frac{3}{4} \frac{\hbar}{\sigma} \tag{24}$$

Therefore, the uncertainty in p, Δp is as follows.

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\sqrt{3}}{2} \frac{\hbar}{\sigma} \tag{25}$$

The uncertainty product is as follows.

$$\Delta x \Delta p = \sqrt{3}\sigma \frac{\sqrt{3}}{2} \frac{\hbar}{\sigma} = \frac{3}{2}\hbar \tag{26}$$

This is more than the minimum uncertainty $\frac{\hbar}{2}$. Therefore this is not a wavefunction with a minimum uncertainty product.

1.4 Superposition of two Gaussians

Now we consider the wave function

$$\psi(x) = A\left(\exp\left(-\frac{\left(x+\frac{a}{2}\right)^2}{4\sigma^2}\right) + \exp\left(-\frac{\left(x+\frac{a}{2}\right)^2}{4\sigma^2}\right)\right) \tag{27}$$

This can be written an equal superposition of two normalized Gaussian wavefunctions.

$$\psi(x) = B(\psi_+(x) + \psi_-(x)) \tag{28}$$

Here $\psi_\pm$ refer to normalized Gaussian wavefunctions centered at $\pm\frac{a}{2}$.

$$\psi_\pm(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x \mp \frac{a}{2})^2}{4\sigma^2}\right) \tag{29}$$

1.4.1 Normalization

Normalization leads to the following condition.

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x |\psi(x)|^2 = 1 \implies |B|^2 \int_{-\infty}^{+\infty} \mathrm{d}x |\psi_+(x) + \psi_-(x)|^2 = 1 \\ \implies |B|^2 \int_{-\infty}^{+\infty} \mathrm{d}x |\psi_+(x)|^2 + |B|^2 \int_{-\infty}^{+\infty} \mathrm{d}x |\psi_-(x)|^2 + 2|B|^2 \int_{-\infty}^{+\infty} \mathrm{d}x \,\mathrm{Re}\,(\psi_+(x)\psi_-(x)) = 1\end{aligned} \tag{30}$$

The integral with each normalized Gaussian wave functions is 1. Therefore the normalization condition is as follows.

$$2|B|^2\left(1 + \int_{-\infty}^{+\infty} \mathrm{d}x \psi_+(x)\psi_-(x)\right) = 1 \tag{31}$$

The cross term is as follows.

$$\psi_+(x)\psi_-(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-\frac{a}{2})^2}{4\sigma^2}\right) \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x+\frac{a}{2})^2}{4\sigma^2}\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x^2 - \frac{a^2}{4})}{2\sigma^2}\right) \tag{32}$$

The integral of the cross term can be evaluated using the integral of a normalized Gaussian.

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x \psi_+(x)\psi_-(x)) &= \int_{-\infty}^{+\infty} \mathrm{d}x \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x^2 - \frac{a^2}{4})}{2\sigma^2}\right) \\ &= \exp\left(-\frac{a^2}{8\sigma^2}\right) \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{x^2}{2\sigma^2}\right) = \exp\left(-\frac{a^2}{8\sigma^2}\right)\end{aligned} \tag{33}$$

Therefore, we can find $|B|$

$$2|B|^2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right) = 1 \implies |B| = \frac{1}{\sqrt{2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}} \tag{34}$$

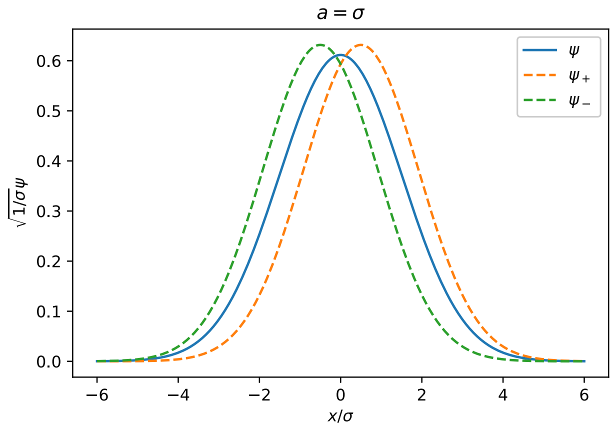
We choose the overall phase to make the wavefunction real. Therefore the normalized wavefunction is the following.

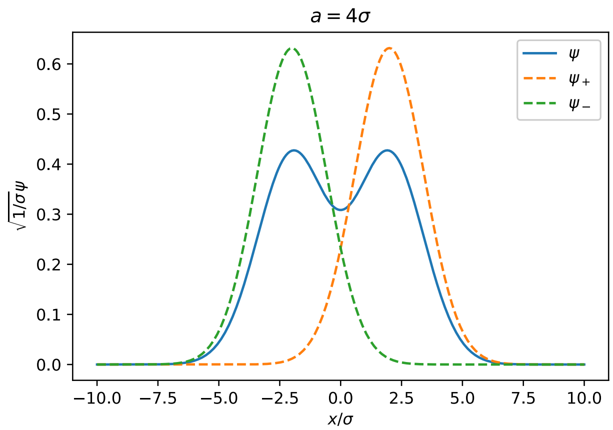
$$\psi(x) = \frac{1}{\sqrt{2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}} \left(\frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-\frac{a}{2})^2}{4\sigma^2}\right) + \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x+\frac{a}{2})^2}{4\sigma^2}\right)\right) \tag{35}$$

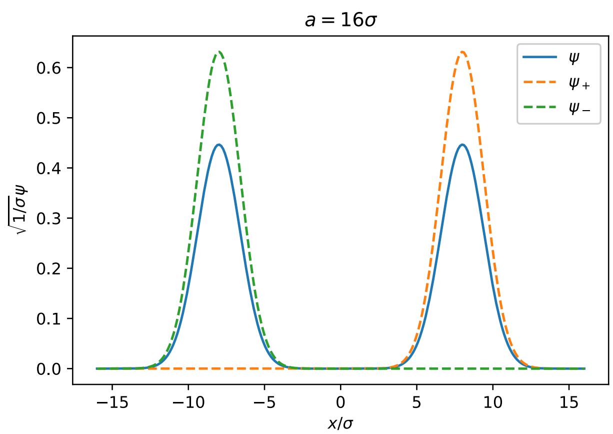
Therefore the normalization constant A is the following.

$$A = \frac{1}{\sqrt{2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}} \frac{1}{(2\pi\sigma^2)^{1/4}} \tag{36}$$

The wave functions are plotted below. When a is large compared σ, the two Gaussian are distinguishable.







1.4.2 Momentum space wave function

The position space wavefunction is an equal superposition of $\psi_+(x)$ and $\psi_-(x)$. We can use the linearity of the Fourier transform to write the momentum space wave function as equal superposition of the Fourier transforms $\tilde{\phi}_+(p)$ and $\tilde{\phi}_-(p)$

$$\tilde{\phi}(p) = B\Big(\tilde{\phi}_+(p) + \tilde{\phi}_-(p)\Big) \tag{37}$$

The functions $\psi_+(x)$ and $\psi_-(x)$ are displaced versions of the Gaussian wavefunction centered at zero, $\psi_{\text{Gauss}}(x)$.

$$\psi_+(x) = \psi_{\text{Gauss}}\big(x - \tfrac{a}{2}\big) \qquad\qquad \psi_-(x) = \psi_{\text{Gauss}}\big(x + \tfrac{a}{2}\big) \tag{38}$$

Therefore, we use the phase shift property of the Fourier transform to find the momentum space wave function.

$$\tilde{\phi}(p) = B\Big(\tilde{\phi}_+(p) + \tilde{\phi}_-(p)\Big) = B\Big(\tilde{\phi}_{\text{Gauss}}(p)\exp\left(+i\frac{pa}{2\hbar}\right) + \tilde{\phi}_{\text{Gauss}}(p)\exp\left(-i\frac{pa}{2\hbar}\right)\Big) = 2B\tilde{\phi}_{\text{Gauss}}(p)\cos\left(\frac{pa}{2\hbar}\right) \tag{39}$$

Recall that for a Gaussian wavefunction with variance in position σ and mean position $x = 0$, the momentum space wavefunction is as follows.

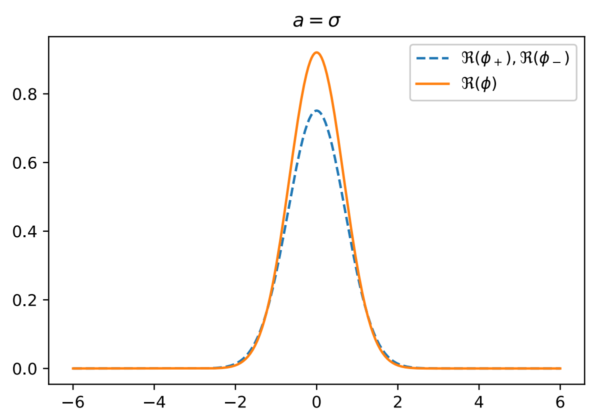
$$\tilde{\phi}_{\text{Gauss}}(p) = \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right) \tag{40}$$

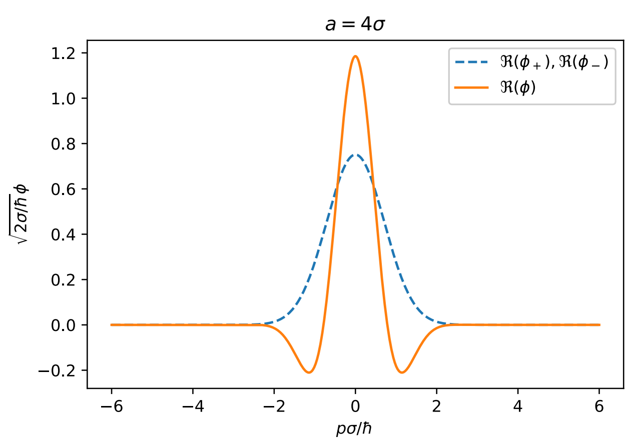
Using this and the expression for B we found earlier, we have the following.

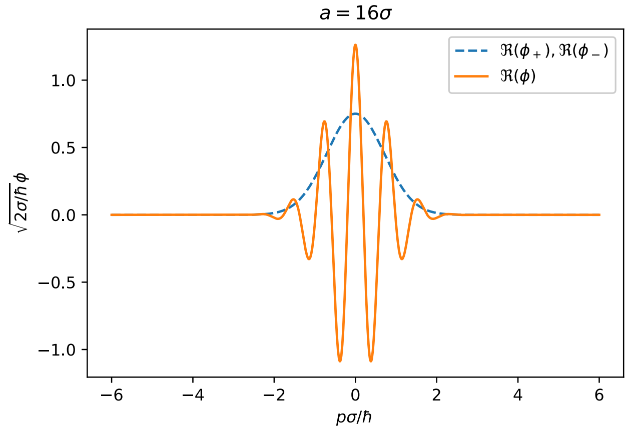
$$\tilde{\phi}(p) = 2\frac{1}{\sqrt{2\Big(1+\exp\left(-\frac{a^2}{8\sigma^2}\right)\Big)}}\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right)\cos\left(\frac{pa}{2\hbar}\right)$$

$$\implies \tilde{\phi}(p) = \frac{1}{\sqrt{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)}}\left(\frac{8\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right)\cos\left(\frac{pa}{2\hbar}\right) \tag{41}$$

These are plotted below. When a is large compared σ, the phase difference between the two Gaussians leads to interference fringes.







1.4.3 Position and Momentum Uncertainties

The position space wave function $\psi(x)$ is even about $x = 0$. Therefore $\langle x \rangle = 0$. Futhermore, the momentum space wave function $\tilde{\phi}(p)$ is even about $p = 0$. Therefore $\langle p \rangle = 0$.

The second moment of position, $\langle x^2 \rangle$ is as follows. We write $|\psi(x)|^2$ using three terms as we did earlier.

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 |\psi(x)|^2 = B^2 \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 |\psi_+(x)|^2 + B^2 \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 |\psi_-(x)|^2 + 2B^2 \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \psi_+(x)\psi_-(x) \tag{42}$$

Using the properties of the Gaussian distribution we noted earlier, the first two integrals become $\sigma^2 + \frac{a^2}{4}$.

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 |\psi_+(x)|^2 = \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \exp\left(-\frac{\left(x - \frac{a}{2}\right)^2}{2\sigma^2}\right) = \sigma^2 + \frac{a^2}{4} \tag{43}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 |\psi_-(x)|^2 = \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \exp\left(-\frac{\left(x + \frac{a}{2}\right)^2}{2\sigma^2}\right) = \sigma^2 + \frac{a^2}{4} \tag{44}$$

The integral of the cross term simplifies as follows.

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \psi_+(x)\psi_-(x) = \exp\left(-\frac{a^2}{8\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) = \exp\left(-\frac{a^2}{8\sigma^2}\right)\sigma^2 \tag{45}$$

Putting them together, we have the following for $\langle x^2 \rangle$.

$$\langle x^2 \rangle = B^2 \left(\left(\sigma^2 + \frac{a^2}{4}\right) + \left(\sigma^2 + \frac{a^2}{4}\right) + 2\exp\left(-\frac{a^2}{8\sigma^2}\right)\sigma^2 \right) = B^2 \left(\left(2 + 2\exp\left(-\frac{a^2}{8\sigma^2}\right)\right) + \frac{a^2}{2} \right) \tag{46}$$

Using the expression for B we found earlier, we get the following.

$$\langle x^2 \rangle = \frac{1}{2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)} \left(\left(2 + 2\exp\left(-\frac{a^2}{8\sigma^2}\right)\right)\sigma^2 + \frac{a^2}{2} \right) = \sigma^2 + \frac{a^2}{4\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)} \tag{47}$$

When $a \gg \sigma$, $\langle x \rangle^2 \to \frac{a^2}{4}$, as expected.
Therefore the uncertainty in position is the following.

$$\Delta x = \sqrt{\sigma^2 + \frac{a^2}{4\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}} = \sigma \sqrt{1 + \frac{a^2}{4\sigma^2\left(1 + \exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}} \tag{48}$$

The calculation for second moment of momentum $\langle p^2 \rangle$ is involved.

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \left| \tilde{\phi}(p) \right|^2 \tag{49}$$

We could write $\tilde{\phi}(p)$ as a superposition of $\tilde{\phi}_+(p)$ and $\tilde{\phi}_-(p)$. However, we do the integral directly for practice. We will see that these are equivalent.

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \left[\frac{1}{\sqrt{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)}} \left(\frac{8\sigma^2}{\pi\hbar^2}\right)^{1/4} \exp\left(-\frac{p^2\sigma^2}{\hbar^2}\right) \cos\left(\frac{pa}{2\hbar}\right) \right]^2 \\
&= \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)} \left(\frac{8\sigma^2}{\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) \cos^2\left(\frac{pa}{2\hbar}\right) \\
&= \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)} \left(\frac{8\sigma^2}{\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) \frac{1}{2}\left(1+\cos\left(\frac{pa}{\hbar}\right)\right) \\
&= \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)} \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/2} \left[\int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) + \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) \cos\left(\frac{pa}{\hbar}\right) \right]
\end{aligned} \tag{50}$$

The first integral is the second moment of momentum for a Gaussian wavefunction in momentum space with zero mean momentum and momentum variance $\frac{\hbar^2}{4\sigma^2}$. This is the term we would get from the integral of $\tilde{\phi}_+(p)$ and $\tilde{\phi}_-(p)$. Therefore $\langle p^2 \rangle$ is the following.

$$\langle p^2 \rangle = \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)} \left[\frac{\hbar^2}{4\sigma^2} + \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) \cos\left(\frac{pa}{\hbar}\right) \right] \tag{51}$$

The integral with the cosine and the Gaussian and the quadratic term is complicated. This integral is of the following form with $A = \frac{2\sigma^2}{\hbar^2}$ and $B = \frac{a}{\hbar}$.

$$\sqrt{\frac{A}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p p^2 e^{-Ap^2} \cos(Bp) = \frac{1}{2}\left(\sqrt{\frac{A}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p p^2 e^{-Ap^2} e^{-iBp} + \sqrt{\frac{A}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p p^2 e^{-Ap^2} e^{+iBp} \right) \tag{52}$$

Each of these integrals can be done as follows.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{d}p p^2 e^{-Ap^2} e^{\pm iBp} = -\frac{\partial}{\partial A}\left(\int_{-\infty}^{+\infty} \mathrm{d}p e^{-Ap^2} e^{\pm iBp} \right) &= -\frac{\partial}{\partial A}\left(\left(\frac{\pi}{A}\right)^{1/2} \exp\left(\frac{-B^2}{4A}\right) \right) \\
&= \frac{\pi^{1/2}}{2A^{5/2}}(A-2B^2) \exp\left(\frac{-B^2}{4A}\right)
\end{aligned} \tag{53}$$

Therefore, our integral is as follows.

$$\begin{aligned}
\sqrt{\frac{A}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p p^2 e^{-Ap^2} \cos(Bp) = \left(\frac{A}{\pi}\right)^{1/2} \frac{\pi^{1/2}}{2A^{5/2}}(A-2B^2) \exp\left(\frac{-B^2}{4A}\right) &= \frac{1}{A^2}(A-2B^2) \exp\left(\frac{-B^2}{4A}\right) \\
&= \left(\frac{1}{2A} - \frac{B^2}{A^2}\right) \exp\left(\frac{-B^2}{4A}\right)
\end{aligned} \tag{54}$$

Plugging A and B back in, we have the following.

$$\begin{aligned}
\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}p\, p^2 \exp\left(-\frac{2p^2\sigma^2}{\hbar^2}\right) \cos\left(\frac{pa}{\hbar}\right) &= \left(\frac{\hbar^2}{4\sigma^2} - \frac{a^2}{\hbar^2}\frac{\hbar^4}{4\sigma^4}\right) \exp\left(-\frac{a^2}{\hbar^2}\frac{\hbar^2}{8\sigma^2}\right) \\
&= \hbar^2\left(\frac{1}{4\sigma^2} - \frac{a^2}{4\sigma^4}\right) \exp\left(-\frac{a^2}{8\sigma^2}\right)
\end{aligned} \tag{55}$$

Therefore the second moment of momentum, $\langle p^2 \rangle$ is as follows.

$$\begin{aligned}\langle p^2 \rangle = \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)}\left[\frac{\hbar^2}{4\sigma^2}+\hbar^2\left(\frac{1}{4\sigma^2}-\frac{a^2}{4\sigma^4}\right)\exp\left(-\frac{a^2}{8\sigma^2}\right)\right] \\ = \frac{1}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)}\left[\frac{\hbar^2}{4\sigma^2}\left(1+\exp\left(-\frac{a^2}{8\sigma^2}\right)\right)-\frac{\hbar^2 a^2}{4\sigma^4}\exp\left(-\frac{a^2}{8\sigma^2}\right)\right] \\ = \frac{\hbar^2}{4\sigma^2}-\frac{\hbar^2 a^2}{4\sigma^4}\frac{\exp\left(-\frac{a^2}{8\sigma^2}\right)}{1+\exp\left(-\frac{a^2}{8\sigma^2}\right)} = \frac{\hbar^2}{4\sigma^2}-\frac{\hbar^2 a^2}{4\sigma^4\left(\exp\left(+\frac{a^2}{8\sigma^2}\right)+1\right)}\end{aligned} \tag{56}$$

Therefore the uncertainty in momentum is as follows.

$$\Delta p = \sqrt{\frac{\hbar^2}{4\sigma^2}-\frac{\hbar^2 a^2}{4\sigma^4\left(\exp\left(+\frac{a^2}{8\sigma^2}\right)+1\right)}} = \frac{\hbar}{2\sigma}\sqrt{1-\frac{a^2}{\sigma^2\left(\exp\left(+\frac{a^2}{8\sigma^2}\right)+1\right)}} \tag{57}$$

When $a \gg \sigma$, the first term is much larger and the uncertainty goes to $\frac{\hbar}{2\sigma}$.
The uncertainty product is

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\left(1+\frac{a^2}{4\sigma^2\left(1+\exp\left(-\frac{a^2}{8\sigma^2}\right)\right)}\right)\left(1-\frac{a^2}{\sigma^2\left(\exp\left(+\frac{a^2}{8\sigma^2}\right)+1\right)}\right)} \tag{58}$$

For $a \ll \sigma$, this approaches $\frac{\hbar}{2}$, the minimum uncertainty. For $a \gg \sigma$, this approaches $\frac{\hbar a}{4\sigma}$, which is much larger than the minimum uncertainty.

Therefore, this wavefunction is not one of minimum uncertainty.