Physics 491: Quantum Mechanics I
Problem Set #4
Due: Wednesday, September 26, 2018

Problem 1: Short answers (10 points)

(a) Which of the following operatorators are *linear* operators (prove explicitly)

(i) $\hat{O}_1\psi(x) = x^3\psi(x)$; (ii) $\hat{O}_2\psi(x) = x\frac{d}{dx}\psi(x)$; (iii) $\hat{O}_3\psi(x) = \lambda\psi^*(x)$; (iv) $\hat{O}_4\psi(x) = e^{\psi(x)}$;

(v) $\hat{O}_5\psi(x) = \frac{d}{dx}\psi(x)$; (vi) $\hat{O}_6\psi(x) = \int_{-\infty}^{\infty} dx'\, x'\psi(x')$;

(b) Consider an electron in a imprenentrable box of size (infinite square well) $a = 1nm$.

(i) What is the energy difference between the ground state and the first excited state (express your answer in eV)?

(ii) Suppose a transition from state *n*=2 to *n*=1 is accompanied by the emission of a photon, as given by the Born rule. What is the wavelength of the emitted photon.

Problem 2: Spreading of a wave packet (25 points)

We saw in P.S.#3 that a localized wave packet in free space will spread due to its initial distribution of momenta, and we estimated the time dependence of the width. This wave phenomenon is known as *dispersion*, arising because the relation between frequency ω and wavenumber k is not linear. Let's revisit this.

Consider a general wave packet in free space at time *t*=0, $\psi(x,0)$.

(a) Show that the wave function at a later time is,

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' K(x,x';t)\,\psi(x')$$

where $K(x,x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im(x-x')^2}{2\hbar t}\right]$ is known as the "propagator".

(Hint. Solve the initial value problem in the usual way -- Decompose $\psi(x,0)$ into the stationary states (here plane waves), add the time dependence, and then re-superpose)

(b) Suppose the initial wave packet is a Gaussian, $\psi(x,0) = \frac{1}{(2\pi a^2)^{1/4}} e^{ik_0 x} e^{-x^2/4a^2}$.

Given the characteristic width a, find the characteristic momentum p_c , energy E_c, and time scale t_c associated with the packet. The time t_c sets the scale at which the packet will spread. Find this for a macroscopic object of mass 1 g and width a =1 cm. Comment.

(c) Show that the wave function as a later time is

$$\psi(x,t) = \frac{1}{\left(2\pi a^2\right)^{1/4}} \frac{1}{\sqrt{1+it/t_c}} \exp\left[\frac{-(x-v_g t)^2}{4a^2} \frac{\left(t_c/t\right)^2}{(1-it_c/t)}\right] \exp\left[i\frac{x^2}{4a^2}\frac{t_c}{t}\right]$$

where $v_g = \frac{\hbar k_0}{m}$ is the group velocity and here $t_c \equiv \frac{2ma^2}{\hbar}$. How does this value of t_c compare to the one you chose in part (b)?

(d) Thus show that the wave packet probability density remains Gaussian with solution

$$P(x,t) \equiv \left|\psi(x,t)\right|^2 = \frac{1}{\sqrt{2\pi a(t)^2}} \exp\left[-\frac{(x-v_g t)^2}{2a(t)^2}\right], \text{ with } a(t) = a\sqrt{1+\left(t/t_c\right)^2}.$$

Compare this to the estimate you made for the square wave packet in P.S. #3

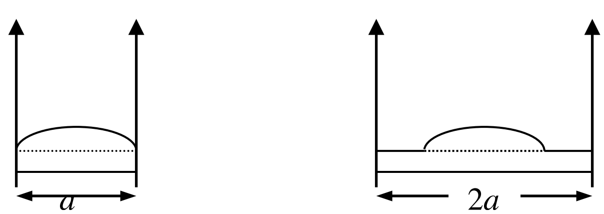
Problem 3: The "Sudden Approximation" (30 points)

Suppose a particle is prepared at t=0 in an infinite square well of width a, $-\frac{a}{2} \le x \le \frac{a}{2}$

$$\psi(x,0_-) = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

The width of the box is *suddenly* expanded to twice its width, $2a$. If this is done very fast, the wave function does *not* have time to change and it is described still as

$$\psi(x,0_+) = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}.$$



However, this wave function is **no longer an eigenstate** of the new Hamiltonian. The goal of this problem is to describe what happens.

(a) Express the initial wave function as a superposition of the new stationary states.
(b) Sketch the probability distribution of possible energies that can be measured.
(c) Find the wave function at all times $\psi(x,t)$.
(d) *Qualitatively*, describe how the wave function evolves.
(e) Extra credit: Use your favorite math package (e.g. Mathematica) to make a movie of the wave function. This will really help your intuition.