Physics 491 - Quantum Mechanics I

Problem Set #4 - Solutions

Prob. 1: Short Answers

(a) Prob 1: Linear Operators:

An operator $\hat{O}: f(x)$ is linear iff

$$\hat{O}: (c_1 f_1(x) + c_2 f_2(x)) = c_1 \hat{O}: f_1(x) + c_2 \hat{O}: f_2(x)$$

- $\hat{O}_1: \psi(x) \equiv x^3 \psi(x)$

$$\hat{O}_1: (c_1 \psi_1(x) + c_2 \psi_2(x)) = x^3 (c_1 \psi_1(x) + c_2 \psi_2(x))$$

$$= c_1 (x^3 \psi_1(x)) + c_2 (x^3 \psi_2(x)) = c_1 \hat{O}_1 \psi_1(x) + c_2 \hat{O}_1 \psi_2(x)$$

$$\Rightarrow \boxed{\hat{O}_1 \text{ is a linear operator}}$$

- $\hat{O}_2: \psi(x) \equiv x \dfrac{d\psi}{dx}$

$$\hat{O}_2: (c_1 \psi_1(x) + c_2 \psi_2(x)) = x \frac{d}{dx} (c_1 \psi_1(x) + c_2 \psi_2(x))$$

$$= c_1 \left(x \frac{d\psi_1}{dx}\right) + c_2 \left(x \frac{d\psi_2}{dx}\right) = c_1 \hat{O}_2 \psi_1(x) + c_2 \hat{O}_2 \psi_2(x)$$

$$\Rightarrow \boxed{\hat{O}_2 \text{ is a linear operator}}$$

- $\hat{O}_3 : \psi(x) \equiv \lambda \psi^*(x)$

$$\hat{O}_3 : (c_1 \psi_1(x) + c_2 \psi_2(x)) = \lambda (c_1^* \psi_1^*(x) + c_2^* \psi_2^*(x))$$

$$= c_1^* (\lambda \psi_1^*(x)) + c_2^* (\lambda \psi_2^*(x)) = c_1^* \hat{O}_3 : \psi_1(x) + c_2^* \hat{O}_3 : \psi_2(x)$$

$$\neq c_1 \hat{O}_3 : \psi_1 + c_2 \hat{O}_3 : \psi_2$$

$\Rightarrow$ $\boxed{\hat{O}_3 \text{ is } \underline{\text{not}} \text{ a linear operator}}$

- $\hat{O}_4 : \psi(x) \equiv e^{\psi(x)}$

$$\hat{O}_4 : (c_1 \psi_1(x) + c_2 \psi_2(x)) = e^{c_1 \psi_1(x) + c_2 \psi_2(x)}$$

$$= e^{c_1 \psi_1(x)} e^{c_2 \psi_2(x)} \neq c_1 \hat{O}_4 \psi_1 + c_2 \hat{O}_4 \psi_2$$

$\Rightarrow$ $\boxed{\hat{O}_4 \text{ is } \underline{\text{not}} \text{ a linear operator}}$

- $\hat{O}_5 : \psi(x) \equiv \frac{d\psi}{dx}$

$$\hat{O}_5 : (c_1 \psi_1(x) + c_2 \psi_2(x)) = \frac{d}{dx}(c_1 \psi_1(x) + c_2 \psi_2(x))$$

$$= c_1 \frac{d\psi_1}{dx} + c_2 \frac{d\psi_2}{dx} = c_1 \hat{O}_5 \psi_1 + c_2 \hat{O}_5 \psi_2$$

$\Rightarrow$ $\boxed{\hat{O}_5 \text{ is a linear operator}}$

$$\hat{O}_6 : \psi(x) = \int_{-\infty}^{x} dx' \left(\psi(x')\, x' \right)$$

$$\hat{O}_6 : \left(c_1 \psi_1(x) + c_2 \psi_2(x) \right) = \int_{-\infty}^{x} dx' \left([c_1 \psi_1(x') + c_2 \psi_2(x')]\, x' \right)$$

$$= c_1 \int_{-\infty}^{x} dx' \psi_1(x')\, x' + c_2 \int_{-\infty}^{x} dx' \, \psi_2(x')\, x')$$

$$= c_1 \, \hat{O}_6 : \psi_1(x) + c_2 \, \hat{O}_6 : \psi_2(x)$$

$\Longrightarrow$ $\boxed{\hat{O}_6 \text{ is a linear operator}}$

(b) Electron, mass $m = 0.9 \times 10^{-30}$ kg in an infinitely high box, dimension $a = 10^{-9}$ m

The energy levels: $E_n = n^2 \dfrac{\pi^2 \hbar^2}{2ma^2} = n^2 \dfrac{\pi^2 (\hbar c)^2}{2 (mc^2) a^2}$

Useful units $\hbar c = 1974$ eV · Å (check this!)

Electron mass $mc^2 = 0.5$ MeV $= 500$ eV

$a = 10^{-9}$ m $= 10$ Å

$\Longrightarrow$ $E_n = n^2 (0.38 \text{ eV})$

- Energy difference $\boxed{E_2 - E_1 = (4-1)\, 0.38\text{eV} = 1.15 \text{ eV}}$

- According to Bohr, the frequency of photon emitted: $\omega = \dfrac{E_2 - E_1}{\hbar} = \dfrac{2\pi c}{\lambda} \Rightarrow \lambda = \dfrac{ch}{E_2 - E_1}$

$\Longrightarrow$ $\boxed{\lambda = 1.07 \, \mu m}$

PROBLEM 2

ⓐ Consider a general wave packet in free space at $t=0$:

(1) $$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b(k)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k;x)\,dk$$

i.e. an infinite superposition of plane waves (free-space stationary states) with arbitrary coefficients, $b(k)$. Including the coefficients $b(k)$;

(2) $$b(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)\,e^{-ikx}\,dx$$ (i.e. Fourier transform)

in the stationary state $\phi(k;x)$ gives the last integral in (1) above

But we know the time dependence of the "stationary states" $\phi(k;x)$ from the TDSE:
$i\hbar\frac{d}{dt}\phi(k;x,t) = \hat{H}\,\phi(k;x,t)$ with $\hat{H} = \frac{\hat{p}^2}{2m}$ for free space. $\Rightarrow$

(3) $$\phi(k;x,t) = \phi(k;x,0)\,e^{-i\omega t}$$

where $\omega \equiv \omega(k)$ is given by $E_k = \hbar\omega(k) = \frac{\hbar^2k^2}{2m}$, $\left(\frac{p^2}{2m}\right.$

Since we know the time dependence of the individual terms in the integral, we just integrate to get the overall time dependence which is then not stationary.

(4) $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int^{\infty} \phi(k;x,t)\,dk = \frac{1}{\sqrt{2\pi}}\int^{\infty} b(k)\,e^{ikx-\omega(k)t}\,dk$$

using (2), being careful with x and x' in (4)

$$b(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x',0)\, e^{-ikx'}\, dx'$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'\, \psi(x',0)\, e^{-ikx'}\, dx' \right\} e^{i(kx-\omega t)}$$

$$= \int_{-\infty}^{\infty} dx'\, \psi(x',0) \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i[\omega t - k(x-x')]}}_{K}$$

Evaluate second integral by completing square with $\omega = \frac{\hbar k^2}{2m}$

$$A^2 k^2 + 2ABk(x-x') + B^2(x-x')^2 = \frac{\hbar t}{2m} k^2 - k(x-x') + B^2(x-x')^2$$

$$A = \sqrt{\frac{\hbar t}{2m}} \qquad 2AB = -1 \qquad B = -\frac{1}{2}\sqrt{\frac{2m}{\hbar t}} \qquad B^2 = \frac{m}{2\hbar t}$$

$$K = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\left[\sqrt{\frac{\hbar t}{2m}}\, k - \sqrt{\frac{m}{2\hbar t}}\,(x-x')\right]^2} e^{i\frac{m}{2\hbar t}(x-x')^2}$$

$$\text{let } u = \sqrt{\frac{\hbar t}{2m}}\, k - \sqrt{\frac{m}{2\hbar t}}\,(x-x')$$

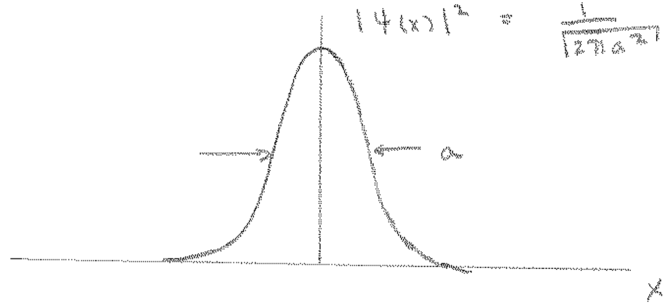
$$du = \sqrt{\frac{\hbar t}{2m}}\, dk$$

$$K = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar t}} \underbrace{\int_{-\infty}^{\infty} du\, e^{-iu^2}}_{\sqrt{\frac{\pi}{i}} \text{ from Schaum's Math Handbook}} \cdot e^{i\frac{m}{2\hbar t}(x-x')^2} = \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\hbar t}}\, e^{i\frac{m}{2\hbar t}(x-x')^2}$$

$$\boxed{K = \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{i\frac{m}{2\hbar t}(x-x')^2} = K(x,x';t) \qquad \text{FREE SPACE PROPAGATOR}}$$

$$\boxed{\psi(x,t) = \int_{-\infty}^{\infty} dx'\, \psi(x',0)\, K(x,x';t)}$$

(b) The probability distribution looks like:



The width a gives us an estimate of Δx at $t=0$. Since this is the only length in the problem, we take it as our "characteristic" length, x_c

$$x_c \equiv a \simeq \Delta x$$

The Heisenberg uncertainty principle gives:

$$\Delta x \Delta p \geq \hbar$$

So taking the minimum momentum uncertainty at $t=0$ for our characteristic momentum:

$$p_c = \Delta p_{min}(t=0) = \frac{\hbar}{\Delta x(t=0)} = \frac{\hbar}{x_c} = \frac{\hbar}{a}$$

And energy: $E_c = \frac{p_c^2}{2m} = \frac{\hbar^2}{2ma^2}$

Associated with any characteristic energy is a characteristic time $t_c \sim \frac{\hbar}{E_c} = \frac{2ma^2}{\hbar}$

Summarizing: $x_c = a \quad p_c = \frac{\hbar}{a} \quad E_c = \frac{\hbar^2}{2ma^2} \quad t_c = \frac{2ma^2}{\hbar}$

For $m = 1g$, $a = 1cm$:

$$t_c = \frac{2(1g)(1cm^2)}{1.05 \times 10^{-34}\ J \cdot s} = \frac{2}{1.05} \frac{10^{-3}\cancel{kg}\ 10^{-4}\cancel{m^2}}{10^{-34}\ \cancel{kg}\frac{\cancel{m^2}}{s}}$$

$$\simeq 2 \times \frac{10^{-7}}{10^{-34}} s$$

$$t_c \simeq 2 \times 10^{27} s$$

much larger than the age of the universe, so we don't expect to see spreading of objects anywhere near macroscopic size.

(c) Given $\psi(x,0) = \frac{1}{(2\pi a^2)^{1/4}} e^{ik_0 x} e^{\frac{-x^2}{4a^2}}$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \, K(x,x';t)\psi(x',0) \, , \qquad K(x,x') = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{im}{2\hbar t}(x-x')^2\right]$$

Let us define $t_c = \frac{2ma^2}{\hbar}$ (characteristic time scale, up to factor of 2)

$$\Rightarrow K(x,x';t) = \frac{1}{\sqrt{4\pi a^2}} \sqrt{\frac{t_c}{it}} \exp\left[i\frac{(x-x')^2}{4a^2}\frac{t_c}{t}\right]$$

$$\Rightarrow \psi(x,t) = \frac{1}{(2\pi a^2)^{1/4}} \frac{1}{\sqrt{4\pi a^2}} \sqrt{\frac{t_c}{it}} \int_{-\infty} dx' \underbrace{\exp\left[i\frac{(x-x')^2}{4a^2}\frac{t_c}{t}\right] \exp\left[\frac{-x'^2}{4a^2} + ik_0 x'\right]}_{\equiv \mathcal{L}}$$

$$\mathcal{L} = e^{i\frac{x^2}{4a^2}\frac{t_c}{t}} \exp\left\{\underbrace{-\underbrace{\frac{1}{4a^2}\left(1 - i\frac{t_c}{t}\right)}_{=A} x'^2 - 2\underbrace{\left[\frac{i}{4a^2}\left(\frac{t_c}{t}\right)x - \frac{ik_0}{2}\right]}_{B} x'}\right\}$$

$$= \exp\left[-A\left(x' - \frac{B}{A}\right)^2\right] e^{\frac{-B^2}{A}}$$

Aside $\int_{-\infty}^{\infty} dx' \, e^{-A\left(x' - \frac{B}{A}\right)^2} = \sqrt{\frac{\pi}{A}}$ when $Re(A) \geq 0$

$$\Rightarrow \psi(x,t) = \frac{1}{(2\pi a^2)^{1/4}} \sqrt{\frac{t_c}{it\left(1 - i\frac{t_c}{t}\right)}} \exp\left\{-\frac{1}{4a^2}\left[\frac{\left(\frac{t_c}{t}\right)x - 2a^2 k_0}{1 - i\frac{t_c}{t}}\right]^2\right\} e^{i\frac{x^2}{4a^2}\frac{t_c}{t}}$$

$$\psi(x,t) = \frac{1}{(2\pi a^2)^{\frac{1}{4}}} \frac{1}{\sqrt{1 + i\frac{t}{t_c}}} \exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x - v_g t)^2}{\left(1 - i\frac{t_c}{t}\right)}\right\} e^{i\frac{x^2}{4a^2}\frac{t_c}{t}}$$

where $v_g = \left.\frac{\hbar k}{m}\right|_{k_0} = \left.\frac{\partial}{\partial k}\omega(k)\right|_{k_0}$

Note: t_c appears just as expected from part (b)

(d) The probability density in position

$$P(x,t) = |\psi(x,t)|^2 = \left| \frac{1}{(2\pi a^2)^{\frac{1}{4}}} \frac{1}{\sqrt{1+i\frac{t}{t_c}}} \exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x-v_g t)^2}{\left(1-i\frac{t_c}{t}\right)}\right\} e^{i\frac{x^2}{4a^2}\frac{t_c}{t}} \right|^2$$

$$= \frac{1}{\sqrt{2\pi a^2}} \left|\frac{1}{\sqrt{1+i\frac{t}{t_c}}}\right|^2 \left|\exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x-v_g t)^2}{\left(1-i\frac{t_c}{t}\right)}\right\}\right|^2 \left|e^{\frac{ix^2}{4a^2}\frac{t_c}{t}}\right|^2$$

Aside: $\left|e^{i\frac{x^2}{4a^2}\frac{t_c}{t}}\right|^2 = 1$ since $|e^{i\phi}| = 1$ for any real ϕ

$$\left|\frac{1}{\sqrt{1+i\frac{t}{t_c}}}\right|^2 = \frac{1}{\sqrt{1+i\frac{t}{t_c}}} \frac{1}{\sqrt{1-i\frac{t}{t_c}}} = \frac{1}{\sqrt{1+\frac{t^2}{t_c^2}}}$$

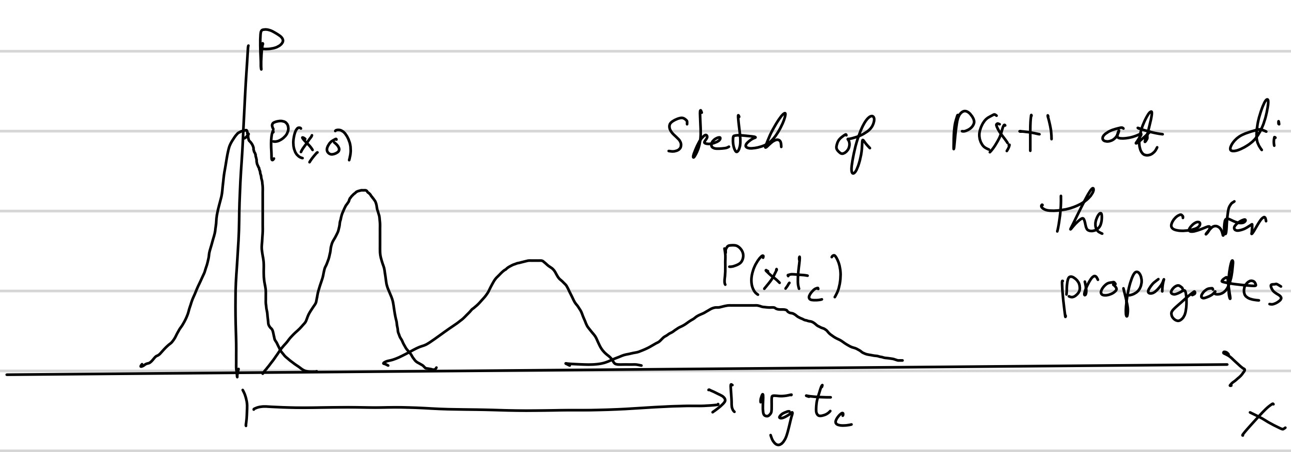
$$\left|\exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x-v_g t)^2}{\left(1-i\frac{t_c}{t}\right)}\right\}\right|^2 = \exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x-v_g t)^2}{\left(1-i\frac{t_c}{t}\right)}\right\} \exp\left\{-\frac{(t_c/t)^2}{4a^2} \frac{(x-v_g t)^2}{\left(1+i\frac{t_c}{t}\right)}\right\}$$

$$= \exp\left\{\frac{-(t_c/t)^2}{4a^2}(x-v_g t)\left[\frac{1}{1-i\frac{t_c}{t}} + \frac{1}{1+i\frac{t_c}{t}}\right]\right\} = \exp\left\{\frac{-(x-v_g t)^2}{2a^2\left(1+\left(\frac{t}{t_c}\right)^2\right)}\right\}$$

Putting it all together,

$$\boxed{P(x,t) = \frac{1}{\sqrt{2\pi a^2(t)}} \exp\left\{\frac{-(x-v_g t)^2}{2a^2(t)}\right\}, \quad a^2(t) = a^2\left(1+\frac{t^2}{t_c^2}\right)}$$

This is exactly what we guessed with basic estimation. The wave packet spreads, and doubles in width in a time $t_c = \frac{2ma^2}{\hbar}$

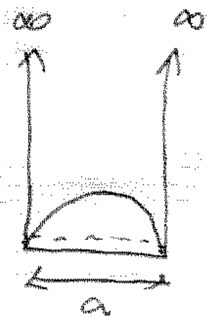


Sketch of $P(x,t)$ at different times

the center of the wave packet propagates at $v_g = \frac{\hbar k_0}{m} = \frac{\langle p\rangle}{m}$

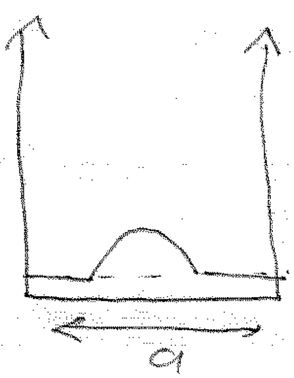
Problem 3: The Sudden Approximation

At $t=0$ a particle is prepared in the ground state of an infinite box



$\Psi(x,0)$

Suddenly, the box is expanded to double its size



$\Psi(x,0_+)$

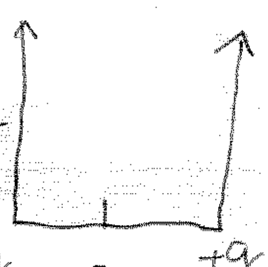
Right after the expansion the wave function is the same.

However, $\Psi(x,0_+)$ is no longer a stationary state. Thus it will evolve with time. The problem is to find $\Psi(x,t)$.

Trick: From the symmetry of the problem, it makes more sense to take the origin at the center of the well

Before

$$u_n = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) & n=1,3,5 \\ \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & n=2,4,6 \end{cases}$$



After $a \to 2a$

$$u'_n(x) = \begin{cases} \sqrt{\frac{1}{a}}\cos\left(\frac{n\pi x}{2a}\right) & n=1,3,5 \\ \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right) & n=2,4,6 \end{cases}$$

(a) The wave function $\psi(x, 0_+)$ can be expanded in terms of the new energy eigenfunctions $\{u_n'(x)\}$

$$\psi(x, 0_+) = \sum_n c_n u_n'(x)$$

where
$$\psi(x, 0_+) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

And
$$c_n = \langle u_n' | \psi \rangle = \int_{-\infty}^{\infty} u_n'^*(x) \, \psi(x, 0_+) \, dx$$

$$= \begin{cases} \int_{-\frac{a}{2}}^{+\frac{a}{2}} dx \left(\frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right)\right)\left(\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)\right) & n = 1, 3, 5, \ldots \\ \int_{-\frac{a}{2}}^{+\frac{a}{2}} dx \left(\frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right)\right)\left(\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)\right) & n = 2, 4, 6, \ldots \end{cases}$$

Now, sine is an odd function over the interval whereas cosine is an even function

$$\Rightarrow \boxed{c_n = 0 \quad \text{for} \quad n = 2, 4, 6 \ldots}$$

For $n = 1, 3, 5, \ldots$

$$c_n = \frac{\sqrt{2}}{a} \int_{-a/2}^{a/2} \cos\left(\frac{n\pi x}{2a}\right) \cos\left(\frac{\pi x}{a}\right) dx$$

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Aside: $\cos\theta_1 \cos\theta_2 = \frac{1}{2}\left[\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)\right]$

$\Rightarrow$ For $n = 1, 3, 5$

$$c_n = \frac{\sqrt{2}}{a}\left(\frac{1}{2}\int_{-\frac{a}{2}}^{a/2} dx \left[\cos\left(\frac{(n+2)\pi x}{2a}\right) + \cos\left(\frac{(n-2)\pi x}{2a}\right]\right)$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+2}\sin\left(\frac{(n+2)\pi x}{2a}\right) + \frac{1}{n-2}\sin\left(\frac{(n-2)\pi x}{2a}\right)\right]_{-a/2}^{a/2}$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{1}{n+2}\sin\left(\frac{(n+2)\pi}{4}\right) + \frac{1}{n-2}\sin\left(\frac{(n-2)\pi}{4}\right)\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{1}{n+2}\cos\left(\frac{n\pi}{4}\right) - \frac{1}{n-2}\cos\left(\frac{n\pi}{4}\right)\right]$$

$$= \frac{-8\sqrt{2}}{\pi}\frac{1}{n^2-4}\cos\left(\frac{n\pi}{4}\right)$$

$$\Rightarrow \boxed{C_n = \begin{cases} \frac{-8\sqrt{2}}{\pi}\frac{1}{n^2-4}\cos\left(\frac{n\pi}{4}\right) & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6 \ldots \end{cases}}$$

$$\boxed{\Psi(x, 0_+) = \sum_{n \text{ odd}} C_n\, u_n'(x)}$$

$$n \text{ odd: } u_n'(x) = \begin{cases} \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi x}{2a}\right) & -a \le x < a \\ 0 & \text{otherwise} \end{cases}$$

(b) The possible energies that can be measured are the energy eigenvalues.

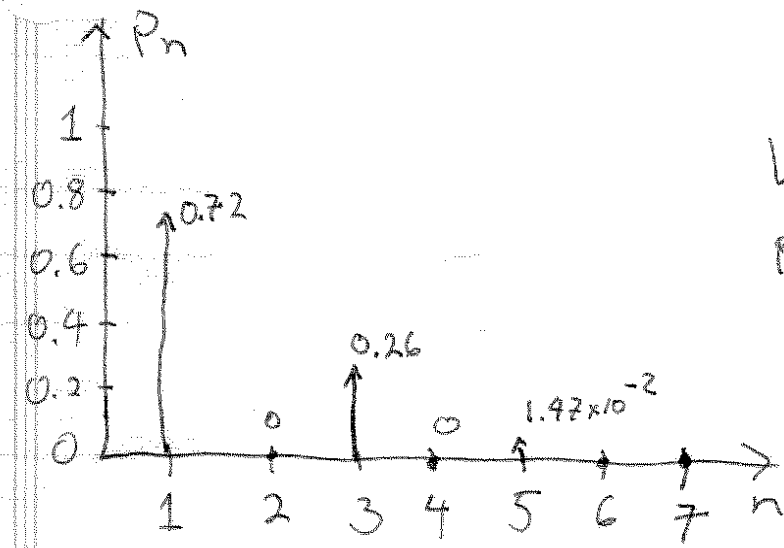
For an infinite box of width $2a$ the eigenvalues are

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2m(2a)^2} = n^2 \frac{\pi^2 \hbar^2}{8ma^2}$$

The ~~prob~~ probability of measuring n^{th} eigenvalue is the square of the probability amplitude

$$\boxed{P_n = |c_n|^2 = \begin{cases} 0 & n = 2, 4, 6 \ldots \\ \frac{64}{\pi^2}\frac{1}{(n^2-4)^2} & n = 1, 3, 5 \ldots \end{cases}}$$

Sketch



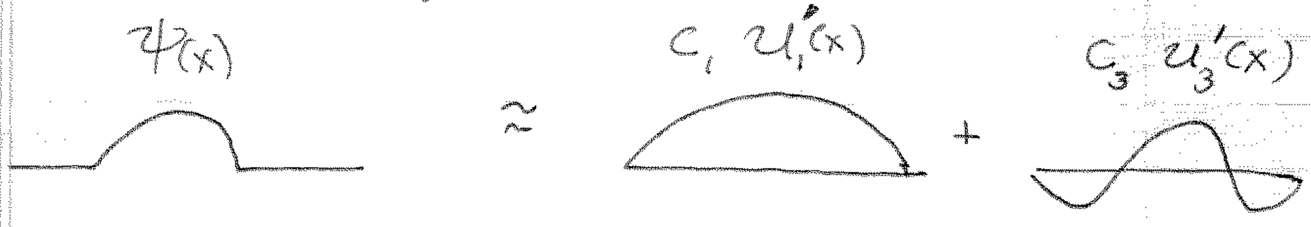
We see that the probability to be in the excited states falls off very rapidly with n

$$P_n \sim \frac{1}{n^4} \quad n \gg 1$$

(c) Given the expansion of the wave function at $t=0$ in terms of stationary state, we know a power series expansion for the solution <u>at all times</u>.

$$\boxed{\begin{gathered}\Psi(x,t) = \sum_{n\ \text{odd}} c_n\, u_n'(x)\, e^{-iE_n t/\hbar} \\ \text{where } c_n = (-1)^{\frac{n+1}{2}} \frac{8}{\pi} \frac{1}{n^2-4} \\ u_n' = \begin{cases} \frac{1}{\sqrt{2}} \cos\left(\frac{n\pi x}{2a}\right) & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}, \quad E_n = n^2 \frac{\pi^2\hbar^2}{8ma^2}\end{gathered}}$$

(d) From our solution in part (b) we see that ~98% of the probabilities is concentrated in the modes $u_1'(x)$ and $u_3'(x)$. This makes sense intuitively



$$\Rightarrow \Psi(x,t) \simeq c_1 u_1'(x)\, e^{-iE_1 t/\hbar} + c_3 u_3'(x)\, e^{-iE_3 t/\hbar}$$

$$= e^{-iE_1 t/\hbar}\left(c_1 u_1'(x) + c_3 u_3'(x)\, e^{-i\Delta E_{31} t/\hbar}\right)$$

where $\Delta E_{31} \equiv E_3 - E_1$

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Now, as in free space, we expect the initially localized wave packet to spread. However, unlike free space, we don't expect this to go indefinitely. In fact, after a finite time, the packet we reconstitute itself because of the discrete nature of spectrum. Under the approximation that only $n=1$ and $n=3$ modes are present, the probability density

$$\boxed{|\Psi(x,t)|^2 \text{ will oscillate at the period } T \cong \frac{2\pi\hbar}{E_{31}} \text{ or } T \approx \frac{\pi}{4E_1}}$$

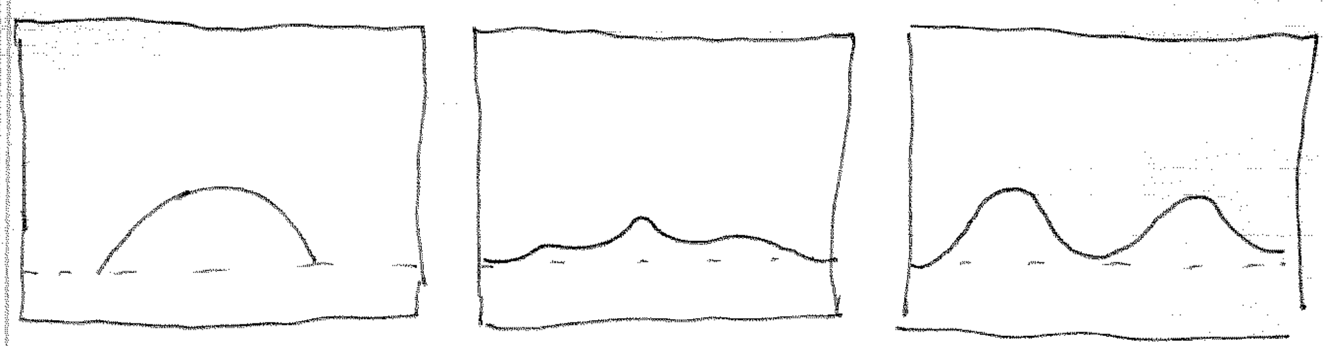
This is not exact, but a good approximation

$$t=0 \qquad |\Psi(x,t)|^2 = |c_1 u_1(x) + c_3 u_3(x)|^2$$

$$t=\frac{T}{2} \qquad |\Psi(x,t)|^2 = |c_1 u_1(x) - c_3 u_3(x)|^2$$

$$t=T \qquad |\Psi(x,t)|^2 = |c_1 u_1(x) + c_3 u_3(x)|^2$$

Sketch (See Mathematica Notebook on web.)



$t=0$ $\qquad$ $t=\frac{T}{4}=\frac{\pi\hbar}{2E_{31}}$ $\qquad$ $t=\frac{T}{2}=\frac{\pi\hbar}{E_{31}}$