Physics 491: Quantum Mechanics I

Problem Set #5
Due: Wednesday, Oct. 24, 2018

Problem 1: Exercises (20 Points)

(a) Consider a Hermitian operator $\hat{A}$ with a complete orthonormal set of eigenvectors satisfying, $\hat{A}|u_a\rangle = a|u_a\rangle$. An arbitrary state can be expand in this basis $|\psi\rangle = \sum_a c_a |u_a\rangle$.

If a measurement of $\hat{A}$ is performed, we find one of the eigenvalues a with probability $P_a = |c_a|^2$. The expectation value is then defined in the usual way $\langle A\rangle = \sum_a a\, P_a$.

Show: $\langle A\rangle = \int dx\, \psi^*(x)\hat{A}\psi(x)$

(b) Prove the following commutation relations:

(i) $\left[\hat{p},\hat{x}^n\right] = -i\hbar n\hat{x}^{n-1}$, (ii) $\left[\hat{p},V(\hat{x})\right] = \frac{\hbar}{i}\left.\frac{dV(x)}{dx}\right|_{x=\hat{x}}$, (ii) $\left[\hat{a},\hat{a}^\dagger\right] = 2\hbar$ (with $\hat{a} = \hat{x} + i\hat{p}$),

Problem 2: Inner products (20 Points)

Consider the Simple Harmonic Oscillator given in the exam. The normalized ground state and first excited state eigenfunctions in the position represenation are

$$\langle x|0\rangle = u_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left\{-\frac{\alpha x^2}{2}\right\}, \quad \langle x|1\rangle = u_1(x) = \left(\frac{4\alpha^3}{\pi}\right)^{1/4} x\exp\left\{-\frac{\alpha x^2}{2}\right\}, \quad \text{where } \alpha = \frac{m\omega}{\hbar}$$

(a) Show that the momentum space representation of the first excited state is

$$\langle p|1\rangle = \tilde{\phi}_1(p) = \frac{-i}{\sqrt{m\omega}} u_1\left(x = \frac{p}{m\omega}\right)$$

(b) Now consider two superpositions: $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$. Show that $\langle\psi_-|\psi_+\rangle = 0$ using:

(i) Energy representation, (ii) Position respresentation, (iii) Momentum Representation

Problem 3: Measurement on a particle in a box (30 points)
Consider a particle in a box of width a, prepared in the ground state.

(a) If one performs a projective (von Neumann) measurement, what are the possible values one can measure for:
(i) energy, (ii) position, (iii) momentum?

(b) What are probabilities for the possible outcomes you found in part (a)?

At some time (call it t=0) we perform measurement of position. However, our detector has only finite resolution and can only resolve the postion into one of three "bins": left, right, and middle. The measurement operators are:

$$\hat{K}_{left} = \int_{-a/2}^{-a/4} dx\,|x\rangle\langle x| \qquad \hat{K}_{middle} = \int_{-a/4}^{+a/4} dx\,|x\rangle\langle x| \qquad \hat{K}_{right} = \int_{+a/4}^{+a/2} dx\,|x\rangle\langle x|$$

Suppose we do a measurement and find the particle in the "middle," i.e., we know the position is for sure in the range $-a/4 < x < a/4$, but are the detector is completely uncertain where it is within this range.

(c1) What is the probability for finding the outcome "middle" given the input state is the ground state of the well.

(c) What is the (normalized) post-measurement wave function?

(d) Immediately after finding the "middle" outcome, if we do a projective measurement, what are the possible values for

(i) energy, (ii) position, (iii) momentum, and with what probabilities?

(e) At later time t, without calculation, describe how you would find the probabilities for a projective measurement of (i) energy, (ii) position, (iii) momentum.