Physics 491: Quantum Mechanics I
Problem Set #6
Due: Wed, Oct. 31, 2018

Problem 1: The Wigner Function (40 points)

For incompatible observables that don't commute, e.g., x and p, there is not a complete set of common eigenvectors. Thus, we cannot assign a joint probability for obtaining x and p at the same time. This contrasts with classical mechanics, where we can define a probability density on phase space $P(x,p)$. How close can we get in quantum mechanics? This problem was first addressed by Eugene Wigner, one of the pioneers on quantum theory, in 1932. He defined what we now call the "Wigner function"

$$W(x,p) = \int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar}\, \psi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)e^{-ipy/\hbar} \text{ , where } \psi(x) \text{ is the wave function.}$$

(a) Show that the Wigner function yields the correct marginals probability densities in x and p,

$$\int_{-\infty}^{\infty} dp\, W(x,p) = \left|\psi(x)\right|^2 = P(x), \quad \int_{-\infty}^{\infty} dx\, W(x,p) = \left|\tilde{\phi}(p)\right|^2 = P(p)\,.$$

(b) For a Gaussian wavepacket, $\psi(x) = \dfrac{1}{(2\pi\Delta x^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\Delta x^2}} e^{ip_0 x/\hbar}$, show that

$$W(x,p) = \frac{1}{\pi\hbar} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} e^{-\frac{(p-p_0)^2}{2\Delta p^2}} \text{ , where } \Delta p = \frac{\hbar}{2\Delta x}$$

(c) Show that the marginals are correct, i.e., part (a) is satisified for this Gaussian wavepacket.

(d) Plot the Wigner function in part (b) as a surface plot. This is an honest to God probability distribution. In this sense, a Gaussian wave packet is "quasiclassical." This is why we we able to understand the spreading of the a Gaussian wave packet in classical terms as a bunch of different classical "runners," running at different speeds. Quantum interference played no role.

(e) Now consider a superposition state of two Gaussian wave packets,

$$\psi(x) = A\left(e^{-(x-x_0)^2/4\sigma_x^2} + e^{-(x+x_0)^2/4\sigma_x^2}\right) \text{, as in Problem 2c of P.S.\#3.}$$

This state is often referred to as a "Schrödinger cat state" when $\sigma >> a$ beause it is a "macroscopic superposition" of distinct positions. Show that the Wigner functions is

$$W_{cat}(x,p) = \frac{A^2}{\pi\hbar}\left(e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} + e^{-\frac{(x+x_0)^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} + 2e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} \cos(4px_0/\hbar) \right), \qquad \sigma_p = \frac{\hbar}{2\sigma_x}$$

(f) (Surface) Plot this Wigner function for $x_0 = \sigma_x,\ 4\sigma_x,\ 8\sigma_x$. Is $W_{cat}(x,p)$ a valid probability distribution?

(g) Calculate the marginals of the Schrödinger-cat Wigner function in x and p as in part (a) and show they are what you expect, given your results from Problem 2c of P.S.#3.