Physics 491: Quantum Mechanics I
Problem Set # 6 Solutions.

Problem:

The Wigner function is defined for a pure state, $\psi(x)$

$$W(x,p) \equiv \int \frac{dy}{2\pi\hbar}\, \psi\left(x+\frac{y}{2}\right) \psi^*\left(x-\frac{y}{2}\right) e^{-\frac{ipy}{\hbar}}$$

The Wigner function is the quantum analog of a joint probability on phase space. Note, $W^*(x,p) = W(x,p)$: the Wigner function is real

In classical statistics, the "marginal" of a joint probability distribution is the probability distribution of only on variable, obtained by summing (integrating) over all values of the other variable.

In Quantum mechanics we have the probability densities separately in position and momentum

$$P(x) = |\psi(x)|^2, \qquad P(p) = |\tilde{\phi}(p)|^2, \quad \text{where} \begin{cases} \tilde{\phi}(p) = \int \frac{dx}{\sqrt{2\pi\hbar}}\, \psi(x)\, e^{-ipx/\hbar} \\ \psi(x) = \int \frac{dp}{\sqrt{2\pi\hbar}}\, \tilde{\phi}(p)\, e^{ipx/\hbar} \end{cases}$$

Let's show that the Wigner function "Marginalizes" correctly

(a) $$\int_{-\infty}^{\infty} dp\, W(x,p) = \int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar}\, \psi\left(x+\frac{y}{2}\right) \psi^*\left(x-\frac{y}{2}\right) \underbrace{\int_{-\infty}^{\infty} dp\, e^{-\frac{ipy}{\hbar}}}_{2\pi\delta\left(\frac{y}{\hbar}\right) = 2\pi\hbar\,\delta(y)}$$

$$= \int_{-\infty}^{\infty} dy\, \psi\left(x+\frac{y}{2}\right) \psi^*\left(x-\frac{y}{2}\right) \delta(y) = |\psi(x)|^2 = P(x) \checkmark$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = \int_{-\infty}^{\infty}\frac{dy}{2\pi\hbar}\underbrace{\int_{-\infty}^{\infty} dx\, \psi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)}_{d}\, e^{-\frac{ipy}{\hbar}}$$

<u>Aside:</u> $d = \int_{-\infty}^{\infty} dx\, \psi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)$

$$= \int_{-\infty}^{\infty} dx \left[\int_{-\infty}^{\infty}\frac{dp_1}{\sqrt{2\pi\hbar}}\, \tilde{\phi}(p_1)\, e^{\frac{ip_1}{\hbar}\left(x+\frac{y}{2}\right)}\right]\left[\int_{-\infty}^{\infty}\frac{dp_2}{\sqrt{2\pi\hbar}}\, \tilde{\phi}(p_2)\, e^{\frac{ip_2}{\hbar}\left(x-\frac{y}{2}\right)}\right]^*$$

$$= \int \frac{dp_1\, dp_2}{2\pi\hbar}\, \tilde{\phi}(p_1)\, \tilde{\phi}^*(p_2)\, e^{i\left(\frac{p_1+p_2}{\hbar}\right)\frac{y}{2}} \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(p_1-p_2)\frac{x}{\hbar}}}_{2\pi\,\delta\left(\frac{p_1-p_2}{\hbar}\right) = 2\pi\hbar\,\delta(p_1-p_2)}$$

$$\Rightarrow d = \int dp_1\, |\tilde{\phi}(p_1)|^2\, e^{ip_1 y/\hbar}$$ (could have seen this from convolution)

$$\therefore \int_{-\infty}^{\infty} dx\, W(x,p) = \int \frac{dp_1}{2\pi\hbar}\, |\tilde{\phi}(p_1)|^2 \underbrace{\int_{-\infty}^{\infty} dy\, e^{i(p_1-p)\frac{y}{\hbar}}}_{2\pi\hbar\,\delta(p_1-p)} = |\tilde{\phi}(p)|^2 = P(p)\ ✓$$

(b) Consider now a Gaussian wavepacket whose (position-space) wavefunction is $\psi(x) = \frac{1}{(2\pi\Delta x^2)^{1/4}}\, e^{-\frac{(x-x_0)^2}{4\Delta x^2}}\, e^{ip_0 x/\hbar}$

From our previous work, $\tilde{\phi}(p) = \frac{1}{(2\pi\Delta p^2)^{1/4}}\, e^{-\frac{(p-p_0)^2}{4\Delta p^2}}\, e^{-ipx_0/\hbar}$

For this wave packet $\langle x\rangle = x_0$, $\langle p\rangle = p_0$ $\quad \Delta p = \frac{\hbar}{2\Delta x}$: minimum uncertainty product.

Now consider the Wigner function

$$W(x,p) = \int \frac{dy}{2\pi\hbar}\, \frac{1}{\sqrt{2\pi\Delta x^2}}\, e^{-\frac{(x-x_0+\frac{y}{2})^2}{4\Delta x^2}}\, e^{-\frac{(x-x_0-\frac{y}{2})^2}{4\Delta x^2}}\, e^{-i(p-p_0)\frac{y}{\hbar}}$$

$$\Rightarrow W(x,p) = \frac{1}{\sqrt{2\pi\Delta x^2}} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} \underbrace{\int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar}\, e^{-\frac{y^2}{8\Delta x^2}} e^{-\frac{i(p-p_0)y}{\hbar}}}_{\frac{1}{2\pi\hbar}\sqrt{8\pi\Delta x^2}\, e^{-\frac{2\Delta x^2 (p-p_0)^2}{\hbar^2}}}$$

(Gaussian integral)

$$\boxed{W(x,p) = \frac{1}{\pi\hbar} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} e^{-\frac{(p-p_0)^2}{2\Delta p^2}} = |\psi(x)|^2 \, |\tilde{\phi}(p)|^2}$$

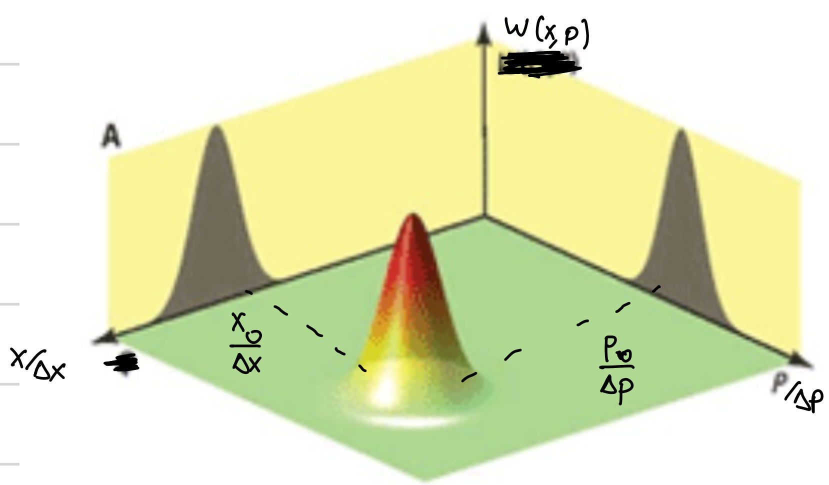
(c) In this case the marginals are simple:

$$\int_{-\infty}^{\infty} dx\, W(x,p) = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 \, |\tilde{\phi}(p)|^2 = |\tilde{\phi}(p)|^2 \checkmark$$

$$\int_{-\infty}^{\infty} dp\, W(x,p) = |\psi(x)|^2 \int_{-\infty}^{\infty} dp\, |\tilde{\phi}(p)|^2 = |\psi(x)|^2 \checkmark$$

(d)

Gaussian Probability Distribution in x and p



Marginals shown as "shadows"

In this plot, the units are chosen dimensionless $\left(\frac{x}{\Delta x}, \frac{p}{\Delta p}\right)$

(e) We now consider a superposition of Gaussian wave packets.

$$\psi(x) = A\left[\frac{e^{-\frac{(x-x_0)^2}{4\sigma_x^2}}}{(2\pi\sigma_x^2)^{1/4}} + \frac{e^{-\frac{(x+x_0)^2}{4\sigma_x^2}}}{(2\pi\sigma_x^2)^{1/4}}\right] = A\left(\psi_+(x) + \psi_-(x)\right), \quad \psi_\pm(x) = \frac{e^{-\frac{(x\mp x_0)^2}{4\sigma_x^2}}}{(2\pi\sigma_x^2)^{1/4}}$$

(Different from stated in problem)

One wave packet is centered at x_0, and the other at $-x_0$. Each has a width σ_x. Note, each packet is real, so $\langle p\rangle = 0$ for each, and $\sigma_p = \frac{\hbar}{2\sigma_x}$.

The Wigner function

$$W(x,p) = \int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar}\, \psi\left(x+\frac{y}{2}\right) \psi^*\left(x-\frac{y}{2}\right) e^{-\frac{ipy}{\hbar}} = A^2\left(W_+(x,p) + W_-(x,p) + 2\,\mathrm{Re}\left(W_{int}(x,p)\right)\right)$$

$W_\pm(x,p) \equiv \frac{1}{\pi\hbar} e^{-\frac{(x\pm x_0)^2}{2\sigma_x^2}}$ is the Wigner function localized at $\pm x_0$

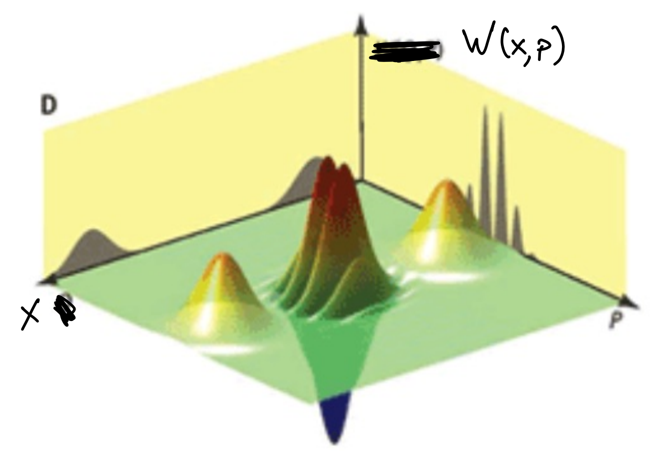
$$W_{int}(x,p) = \int_{-\infty}^{+\infty} \frac{dy}{2\pi\hbar}\, \psi_+\left(x+\frac{y}{2}\right) \psi_-^*\left(x-\frac{y}{2}\right) e^{-\frac{ipy}{\hbar}} = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\infty} \frac{dy}{2\pi\hbar}\, e^{-\frac{(x-x_0+\frac{y}{2})^2}{4\sigma_x^2}} e^{-\frac{(x+x_0-\frac{y}{2})^2}{4\sigma_x^2}} e^{-\frac{ipy}{\hbar}}$$

$$\Rightarrow W_{int} = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{x^2}{2\sigma_x^2}} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-\frac{(y-2x_0)^2}{8\sigma^2}} e^{-\frac{ipy}{\hbar}}\, dy$$

$$\sqrt{8\pi\sigma_x^2}\, e^{-\frac{2p^2\sigma_x^2}{\hbar^2}} e^{-2ipx_0/\hbar}$$ (Gaussian integral)

$$= \frac{1}{\pi\hbar} e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} e^{2ipx_0/\hbar}$$

$$\Rightarrow \boxed{W(x,p) = \frac{A^2}{\pi\hbar}\left[e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} + e^{-\frac{(x+x_0)^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} + 2\, e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{p^2}{2\sigma_p^2}} \cos\left(\frac{2px_0}{\hbar}\right)\right]}$$



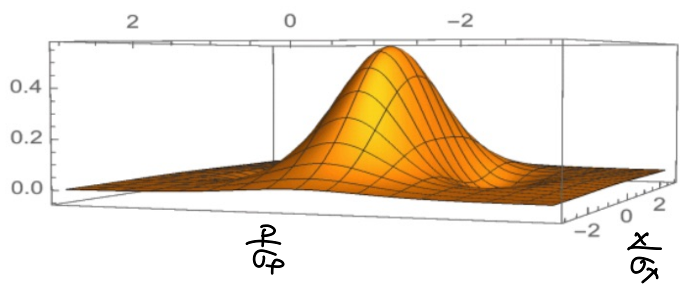
This is a sketch of the Wigner function for the Schrödinger Cat. The interference between the two wave packets appear as "fringes"

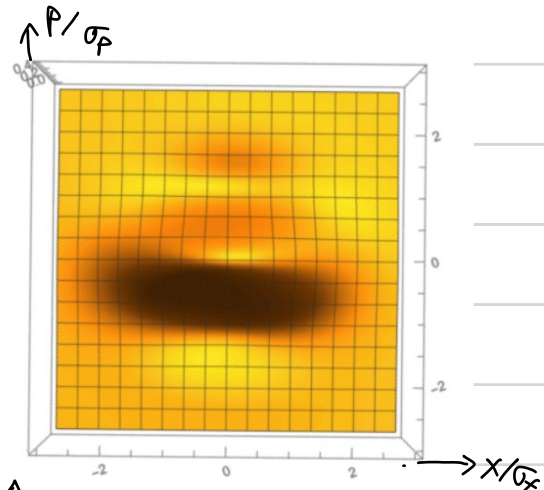
Credit: P. Grangier, "Make It Quantum and Continuous", Science (Perspective) 332, 313 (2011)

This is not a proper probability distribution because the Wigner function can be negative. Thus, the Wigner function is known as as "quasi" probability distribution. Negativity in the Wigner function is a signature on nonclassicality.

Plots shown in dimensionless units: $\frac{x}{\sigma_x}$, $\frac{p}{\sigma_p}$

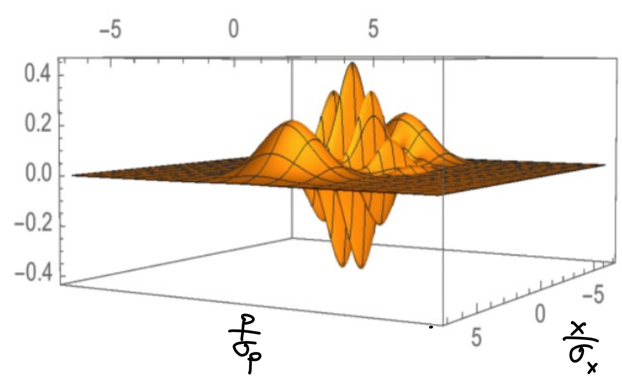
$x_0 = \sigma_x$

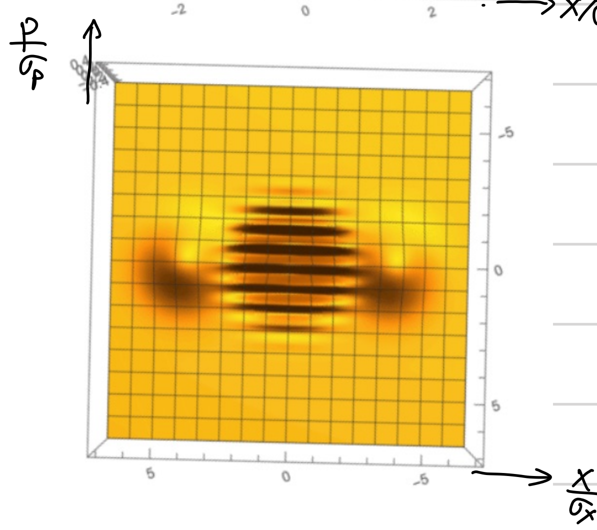




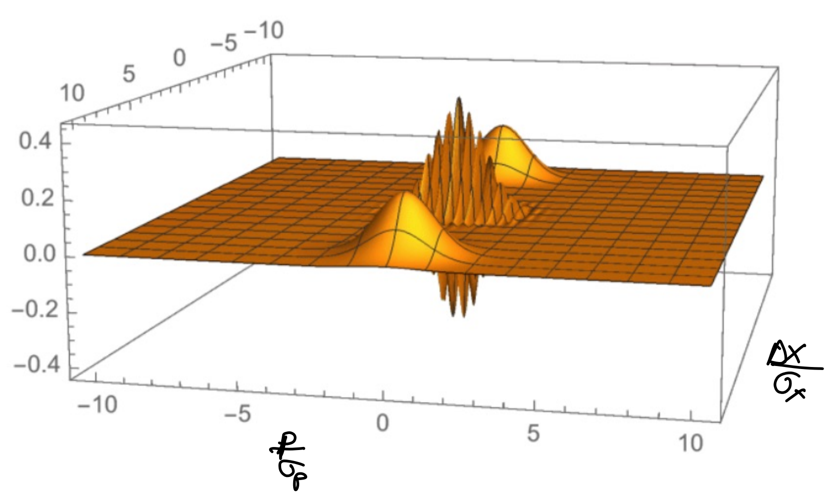
$_0 = 4\sigma_x$

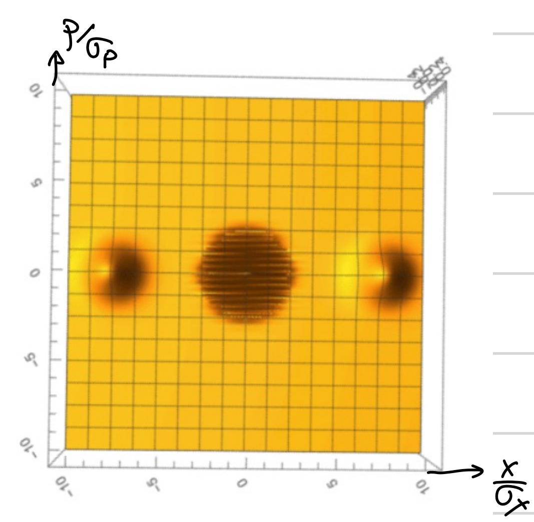
x





$x_0 = 8\sigma_x$





Notes: (1) The Wigner function shows <u>negativity</u> associated with quantum interference.

(2) The more "macroscopic," i.e. the larger the separation of the wave packets compared to the width of the packets, the larger the oscillations

(3) A "classical" probability distribution would be $W(x,p) = W_+(x,p) + W_-(x,p)$, i.e. the particle has a probability to be near x_0 or $-x_0$. The oscillations are <u>nonclassical</u>. These are the "whiskers" of Schrödinger's Cat!

(g) Marginals

$$\int_{-\infty}^{\infty} dp\, W(x,p) = \frac{A^2}{\pi\hbar}\left[e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} \sqrt{2\pi\sigma_p^2} + e^{-\frac{(x+x_0)^2}{2\sigma_x^2}} \sqrt{2\pi\sigma_p^2} + 2 e^{-\frac{x^2}{2\sigma_x^2}} \underbrace{\int_{-\infty}^{\infty} dp\, e^{-\frac{p^2}{2\sigma_p^2}} \cos\left(\frac{2px_0}{\hbar}\right)}_{\sqrt{2\pi\sigma_p^2}\, e^{-2\sigma_p^2 x_0^2/\hbar^2}} \right]$$

$$= A^2 \left[\frac{e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}}{\sqrt{2\pi\sigma_x^2}} + \frac{e^{-\frac{(x+x_0)^2}{2\sigma_x^2}}}{\sqrt{2\pi\sigma_x^2}} + 2\, \frac{e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{x_0^2}{2\sigma_x^2}}}{\sqrt{2\pi\sigma_x^2}} \right]$$

$$= A^2 \left[|\psi_+(x)|^2 + |\psi_-(x)|^2 + 2\,\mathrm{Re}\left(\psi_+(x)\, \psi_-^*(x)\right) \right] = |\psi(x)|^2 \checkmark$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = \frac{A^2}{\pi\hbar}\left[\sqrt{2\pi\sigma_x^2}\, e^{-\frac{p^2}{2\sigma_p^2}} + \sqrt{2\pi\sigma_x^2}\, e^{-\frac{p^2}{2\sigma_p^2}} + 2\sqrt{2\pi\sigma_x^2}\, e^{-\frac{p^2}{2\sigma_p^2}} \cos\left(\frac{2px_0}{\hbar}\right) \right]$$

$$= \frac{A^2}{\sqrt{2\pi\sigma_p^2}} \left(1 + \cos\left(\frac{2px_0}{\hbar}\right)\right) e^{-\frac{p^2}{2\sigma_p^2}} = 4A^2 \underbrace{\cos\left(\frac{px_0}{2}\right)}_{\uparrow} \frac{e^{-\frac{p^2}{2\sigma_p^2}}}{\sqrt{2\pi\sigma_p^2}}$$

interference fringes!

$$= |\tilde{\phi}(p)|^2 \checkmark$$

where $\tilde{\phi}(p) = A\left(\tilde{\phi}_+(p) + \tilde{\phi}_-(p)\right) = A\left[\frac{e^{-\frac{p^2}{4\sigma_p^2}}}{(2\pi\sigma_p^2)^{1/4}} e^{ipx_0/\hbar} + \frac{e^{-\frac{p^2}{4\sigma_p^2}}}{(2\pi\sigma_p^2)^{1/4}} e^{-ipx_0/\hbar} \right]$

$$= 2A \cos\left(\frac{px_0}{\hbar}\right) \frac{e^{-\frac{p^2}{4\sigma_p^2}}}{(2\pi\sigma_p^2)^{1/4}}$$