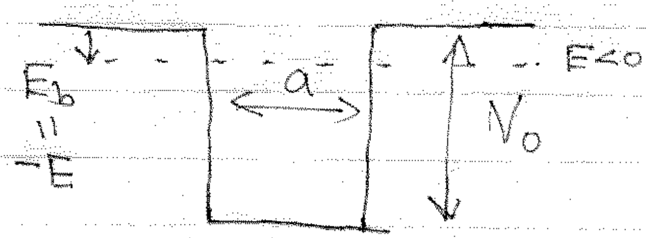
Physics 491 - Quantum I

Problem Set 7 - Solutions

Problem 1: Finite Square Well



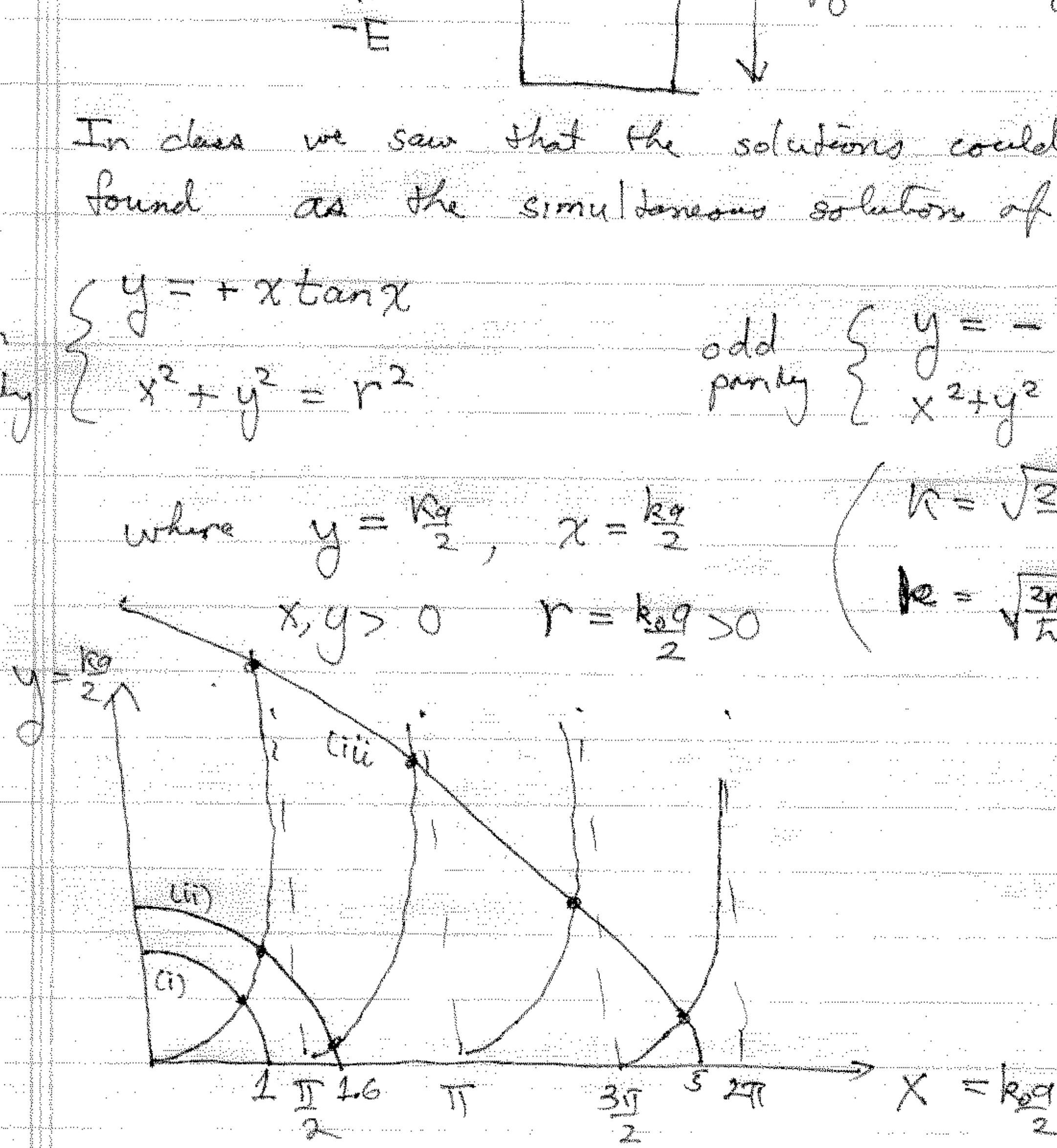
$$k_0 \equiv \sqrt{\frac{2mV_0}{\hbar^2}}$$

In class we saw that the solutions could be found as the simultaneous solutions of:

even parity $\begin{cases} y = +x\tan x \\ x^2 + y^2 = r^2 \end{cases}$ odd parity $\begin{cases} y = -x\cot x \\ x^2 + y^2 = r^2 \end{cases}$

where $y = \frac{\kappa a}{2}$, $x = \frac{ka}{2}$ $\left(\kappa = \sqrt{\frac{2mE_b}{\hbar^2}} \quad k = \sqrt{\frac{2m}{\hbar^2}(V_0 - E_b)} \right)$

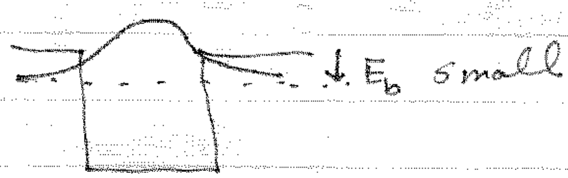
$x, y > 0 \quad r = \frac{k_0 a}{2} > 0$



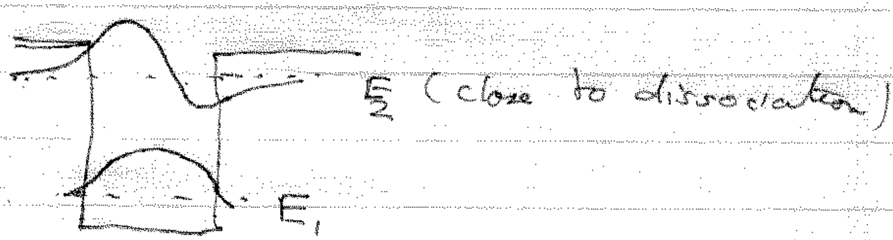
Shown above are the solutions for three different choices of $\frac{k_0 a}{2} = r$

(a)

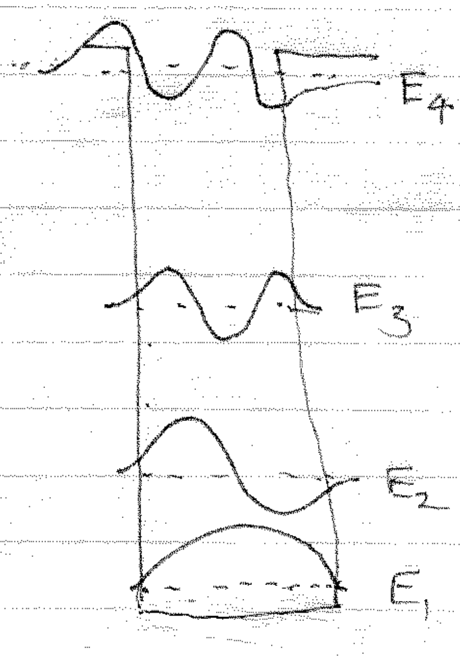
Case (i) $\frac{k_0 a}{2} = 1 \Rightarrow$ One bound state



Case (ii) $\frac{k_0 a}{2} = 1.6 \Rightarrow$ Two bound states

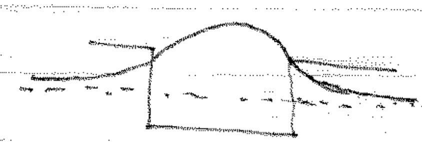


Case (iii) $\frac{k_0 a}{2} = 5 \Rightarrow$ 4 bound states



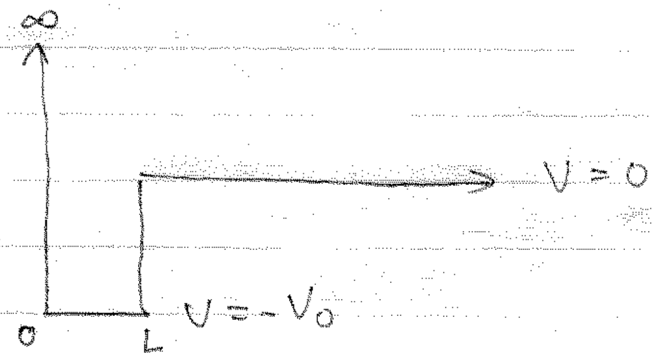
Note: The states deep in the well (i.e. the ground and first excited state) are close to the infinite well solutions.

(b) From the graphical solution we see that there is always one solution for $0 < \frac{k_0 a}{2} < \frac{\pi}{2}$. This is an even parity solution:

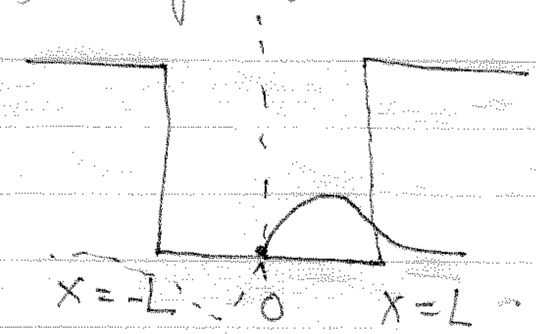


$\Leftarrow$ Single bound state

(c) Consider the half-infinite / half-finite well



The stationary states must satisfy the boundary condition $u(0) = 0$. But the odd parity solutions of the well,



Satisfy all the proper boundary conditions.

Thus <u>Odd parity</u> solution of well $a = 2L$ have the same spectrum ~~of~~ as our half-infinite, half-finite well

The odd parity solution requires $\frac{k_0 a}{2} > \frac{\pi}{2}$

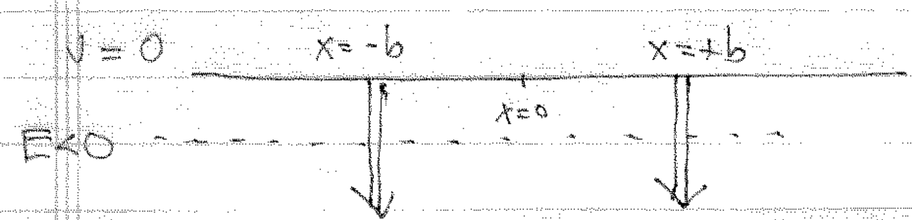
$$\Longrightarrow \boxed{k_0 L > \frac{\pi}{2}}$$

using $\frac{ka}{2} = kL$

(d) General odd parity case: $\boxed{K = -k\cot(kL)}$

Problem 2: Double Well of Delta functions

We consider a potential $V(x) = -U_0\,\delta(x+b) - U_0\,\delta(x-b)$



Two very deep and narrow potential wells

This potential is invariant under parity $V(x) = V(-x)$

$\Rightarrow$ Stationary states are eigenstates of parity

(a) Even parity $u(x) = u(-x)$

We seek solutions of the T.I.S.E. for $E < 0$.

Thus over all space (except at $x = \pm b$) we are in the classically forbidden region, and since V is constant the the solutions are of the form $e^{\pm \kappa x}$

where $\kappa = \sqrt{\frac{2m}{\hbar^2}(\underset{0}{\underset{\shortparallel}{V}} - E)} = \sqrt{\frac{2m}{\hbar^2}\underset{E_b}{\underset{\shortparallel}{(-E)}}} = \sqrt{\frac{2m}{\hbar^2}E_b}$

Also, we must have $u(x) \to 0$ at $x = \pm\infty$.

Thus the even parity solutions have the form

$$u_{even}(x) = \begin{cases} B e^{+\kappa x} & x < -b \\ A(e^{\kappa x} + e^{-\kappa x}) & -b < x < b \\ B e^{-\kappa x} & x > b \end{cases}$$

(b) We now use the boundary conditions across the delta function at $x = b$ (nothing new at $x = -b$ by parity)

$u(x)$ continuous $\Rightarrow$ $A(e^{\kappa b} + e^{-\kappa b}) = B e^{-\kappa b}$

$\frac{du}{dx}$ discontinuous : $\left.\frac{du}{dx}\right|_{b-\epsilon} - \left.\frac{du}{dx}\right|_{b+\epsilon} = \frac{2m}{\hbar^2} U_0\, u(b)$

$$\Rightarrow \kappa A(e^{\kappa b} - e^{-\kappa b}) - (-\kappa B e^{-\kappa b}) = \frac{2m}{\hbar^2} U_0 B e^{-\kappa b}$$

Substitute in $A(e^{\kappa b} + e^{-\kappa b})$ for $B e^{-\kappa b}$

$$\Rightarrow 2\kappa A e^{\kappa b} = \underbrace{\frac{2m}{\hbar^2} U_0}_{2\kappa_0} A(e^{\kappa b} + e^{-\kappa b})$$

$$\Rightarrow e^{\kappa b} = \frac{\kappa_0}{\kappa}(e^{\kappa b} + e^{-\kappa b})$$

$$\Rightarrow 1 = \frac{\kappa_0}{\kappa}(1 + e^{-2\kappa b})$$

$$\Rightarrow \boxed{\frac{\kappa}{\kappa_0} = 1 + e^{-2\kappa b}}$$ even parity

(c) For odd parity we must have

$$u_{odd}(-x) = -u_{odd}(-x)$$

the nature of the solutions is the same, classically forbidden everywhere, $u = e^{\pm \kappa x}$

Thus,

$$u_{odd}(x) = \begin{cases} -Be^{+\kappa x} & x < -b \\ A\,(e^{\kappa x} - e^{-\kappa x}) & -b < x < b \\ Be^{-\kappa x} & x > b \end{cases}$$

(d) Following the same procedure:

$u(x)$ continuous: $A(e^{\kappa b} - e^{-\kappa b}) = Be^{-\kappa b}$

$$\Delta \left.\frac{du}{dx}\right|_b = 2\kappa_0 u(b) \Rightarrow \kappa A(e^{\kappa b} + e^{-\kappa b}) - (-\kappa Be^{-\kappa b}) = 2\kappa_0 B e^{-\kappa b}$$

$$\Rightarrow 2\kappa A e^{\kappa b} = 2\kappa_0 A(e^{\kappa b} - e^{-\kappa b})$$

$$\Rightarrow \boxed{\frac{\kappa}{\kappa_0} = 1 - e^{-2\kappa b}} \quad \text{odd parity}$$

(e) When $b \to 0$

- even parity: $\frac{\kappa}{\kappa_0} \to 2 \Rightarrow \kappa \to 2\kappa_0$

$$E_b = \frac{(\hbar\kappa)^2}{2m} = 4\,\frac{(\hbar\kappa_0)^2}{2m}$$

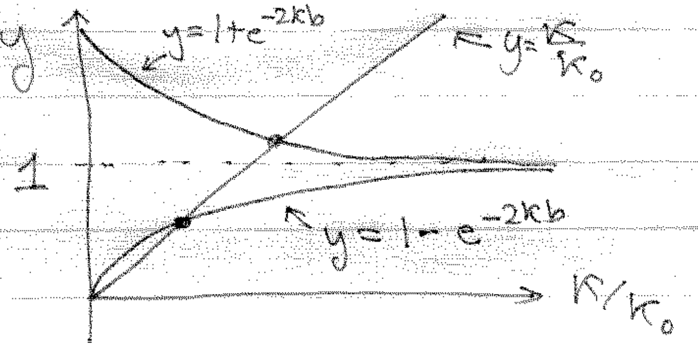
- odd parity: $\frac{\kappa}{\kappa_0} \to 0 \Rightarrow \kappa \to 0$

No bound state

Thus, when $b \to 0$, there is only <u>one bound state</u> (even parity) as expected. The reason the binding energy is 4 times larger than that of the single delta function is that the binding energy goes like the square area inside the equivalent finite well, which is 4 times larger.

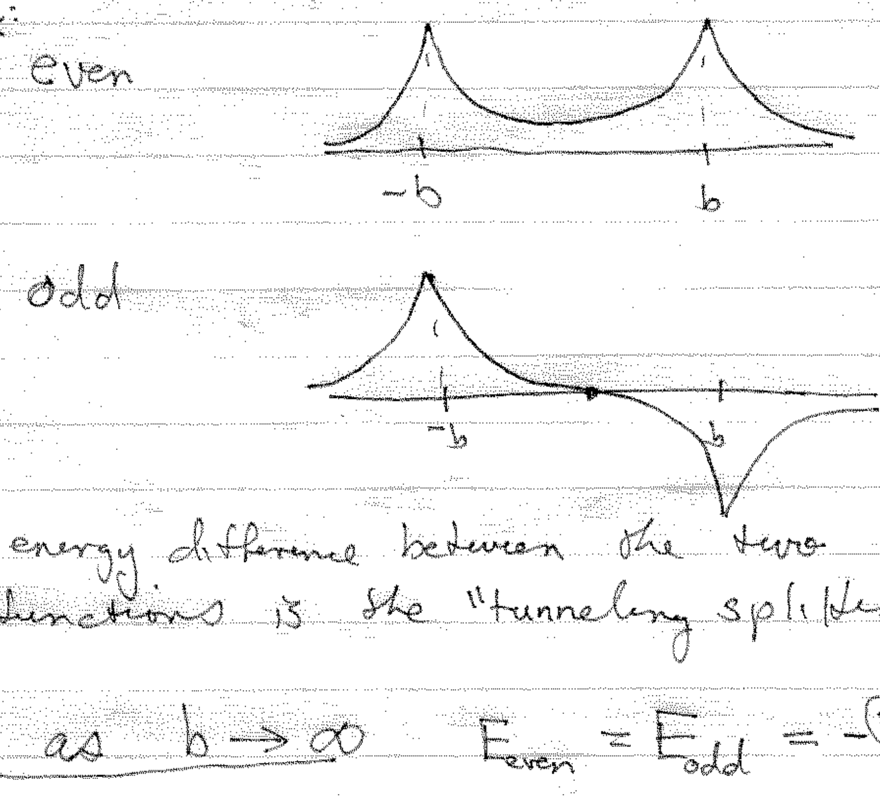
(f) We seek solutions of $\frac{\kappa}{\kappa_0} = 1 \pm e^{-2\kappa b}$ ← even parity (+), odd parity (−)

Graph both sides w.r.t. $\frac{\kappa}{\kappa_0}$



The dots are the solutions

Sketch:

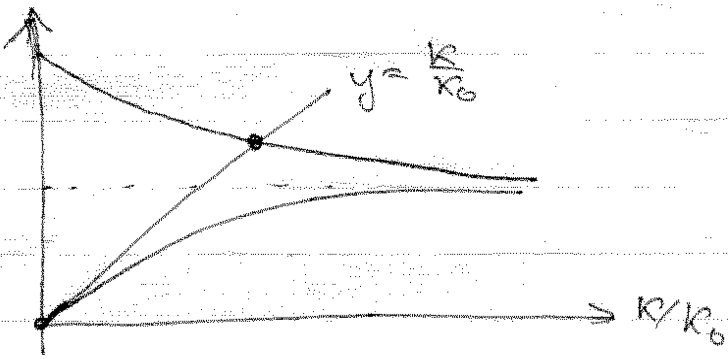


The energy difference between the two eigenfunctions is the "tunneling splitting"

Note as $b \to \infty$ $\quad E_{even} = E_{odd} = -\frac{(\hbar\kappa_0)^2}{2m} = -\frac{mV_0^2}{2\hbar^2}$

The even and odd solutions become degenerate corresponding to ~~two~~ independent and completely ~~[illegible]~~ separate delta functions, each with its own bound state.

(g) Now there is a critical value of b for which the odd parity solution becomes unbound



When the slope of the lower curve (the odd parity case) is ≤ 1 (at the origin), the the are only even parity solutions:

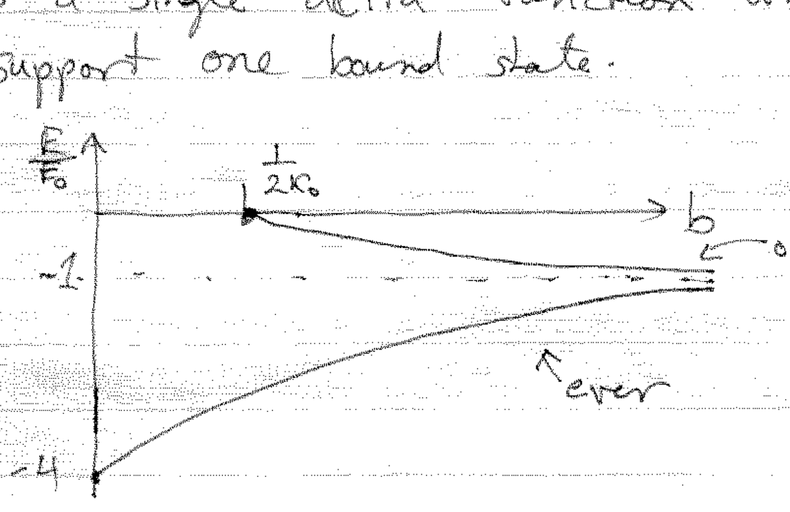
$\Rightarrow$ Let $x = \frac{\kappa}{\kappa_0}$ $\quad f(x) = 1 - e^{-(2\kappa_0 b)x}$

$$\left.\frac{df}{dx}\right|_0 = 2\kappa_0 b$$

$\therefore$ Only one bound state when $\boxed{b < \frac{1}{2\kappa_0}}$

We know at some point the odd parity solution must become unbound since at $b \to 0$ there is a single delta function which can only support one bound state.

Sketch



$E_0 = \frac{(\hbar\kappa_0)^2}{2m}$