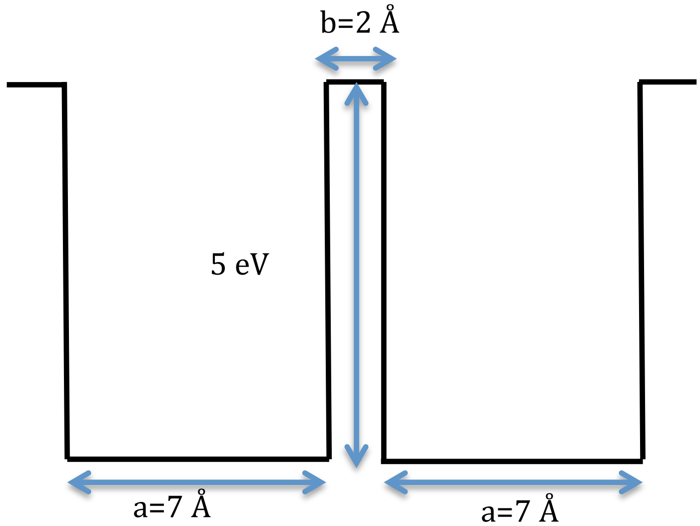
Physics 491: Quantum Mechanics
Problem Set #8
Due: Wed, Nov. 14, 2018

Problem 1: Tunnelling in multiple wells (25 points)

Consider a finite double well for an electron. All energy units are in eV and length in Å.



If the barrier between the wells were infinitely thick, the spectrum of bound states would be doubly degenerate. Due to tunnelling, the energy of these doubly degenerate states is split. Using some advanced techniques, one can show that the "tunnelling splitting" for the n^{th} level is approximately, $|u_n\rangle$

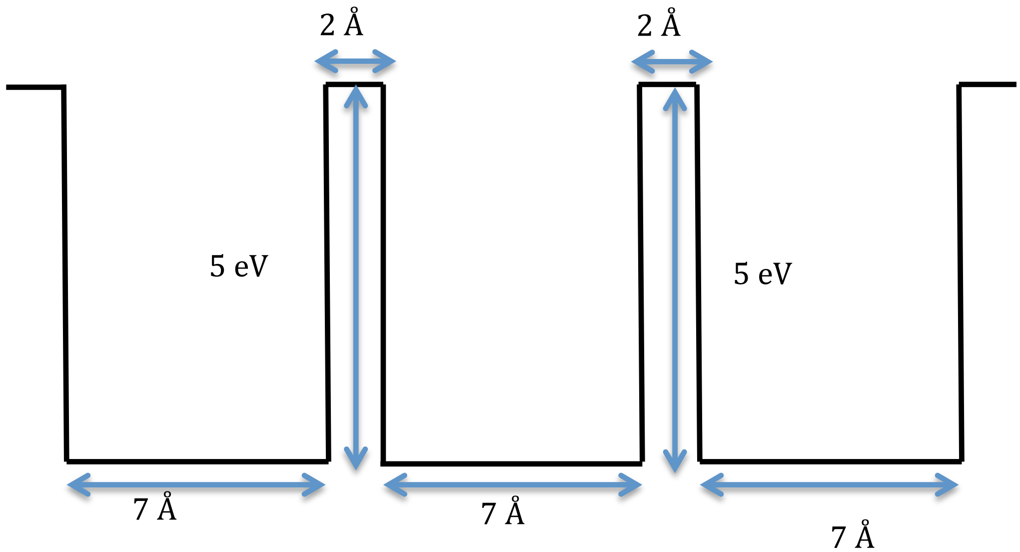
$$|\Delta E_n| = |E_n| e^{-\kappa_n b}, \quad |E_n| = |V_0 - E_{b,n}|$$

where κ_n is the imaginary wavenumber insider the barrier for the n^{th} (nontunneling) level, and $E_{b,n}$ is the binding energy for the n^{th} (nontunneling) level.

(a) Using the knowledge of the solution to the finite well, estimate and sketch the energy level diagram for the bound states.

(b) Sketch all of the bound state wave functions.

Now consider a triple well (next page)



(c) Without calculation, sketch the energy level diagram.

Because this potential is reflection-symmetric and there is no degneracy, the bound states are eigenstates of parity. The previously degenerate triple of bound states for an infinite barrier are "tunnel split". The triple of states for the n^{th} level can be approximated as

$$|u_{n,1}\rangle = \frac{1}{\sqrt{2}}\left(|u_{n,left}\rangle - |u_{n,right}\rangle\right),\ |u_{n,2}\rangle = \frac{1}{\sqrt{3}}\left(|u_{n,left}\rangle + |u_{n,mid}\rangle + |u_{n,right}\rangle\right),$$

$|u_{n,3}\rangle = \frac{1}{\sqrt{6}}\left(|u_{n,left}\rangle - 2|u_{n,mid}\rangle + |u_{n,right}\rangle\right)$ where $|u_{n,left}\rangle, |u_{n,mid}\rangle, |u_{n,right}\rangle$ are the (nontunnelling) n^{th} bound states of of the left, middle, and right wells respectively.

(d) Put these in energy eigenstates in order and sketch the wavefunctions for n=1 and n=2.

(e) Extra Credit: Now consider a periodic potential of an infinite number of wells. Argue how the energy levels consists of a series of continuous "energy bands" separated band gaps.

Problem 2: Scattering from Delta Function Barriers (20 points)

Consider a *repulsive* delta function potential, $V(x) = +U_0\delta(x)$, with $U_0 > 0$.

(a) Given a plane wave incident from the left at energy *E*, show that the reflection and transmission *amplitudes* are

$$r = \sqrt{R}\, e^{i\phi_r}\text{, and } t = i\sqrt{T}\, e^{i\phi_r}\text{,}$$

$$\text{where } R = \frac{\kappa^2}{k^2+\kappa^2},\ T = \frac{k^2}{k^2+\kappa^2},\ \tan\phi_r = \frac{-\sqrt{T}}{-\sqrt{R}}\text{, with } k = \sqrt{\frac{2m}{\hbar^2}E},\ \kappa = \frac{mU_0}{\hbar^2}.$$

(b) Now consider two barriers, described by the potential $V(x) = +U_0\delta(x) + U_0\delta(x-L)$. Given an incident plane wave on the left, find the total amplitude reflection coefficient for waves traveling x<0, and the amplitude transmission coefficient for waves traveling in the region $x > L$.

(c) Plot $R_{total} = |r_{total}|^2$ and $T_{total} = |t_{total}|^2$. Show that $R_{total} = 0$ and $T_{total} = 1$ when $kL + \phi_r = n\pi$. This is an example of a *scattering resonance* when the energy of the incident particle matches a "quasibound state" of the potential. Please comment.