**Physics 491: Quantum Mechanics II**
**Problem Set #10**
**Due: Thursday, Nov. 29, 2018 @5:00PM in TA Mailbox**

**Extra Credit 10 points:**
(a) Consider the SHO where the system intially prepared in a state $|\psi(0)\rangle$, whose wave function is $\psi(x,0) = u_0(x - x_0)$, i.e. the ground state wave function displaced from the origin, and centered at $x_0$ . Show that

$$\psi(x,0) = \sum_{n-0}^{\infty} c_n u_n(x) \text{ , where } c_n = e^{-\frac{X_0^2}{2}} \frac{X_0^n}{\sqrt{n!}} \text{ , } X_0 = \frac{x_0}{x_c} \text{ , } x_c = \sqrt{\frac{2\hbar}{m\omega}} \text{ .}$$

Hint: Use the generating function for Hermite polynomials $\sum_{n=0}^{\infty} H_n(z) \frac{y^n}{n!} = \exp\left(2zy - y^2\right)$

**Extra Credit 5 points:**
(b) Show that at time $t$=0, the probabilitity to find the particle in the $n^{th}$ excited state is

$$P_n = e^{-\langle \hat{N} \rangle} \frac{\langle \hat{N} \rangle^n}{n!}$$ . This is known as a Poisson distribution.

**Extra Credit 10 points:**
(c) Show that at a later time, the wave function (up to an negligable phase, independent of $x$) is

$$\psi(x,t) = u_0(x - x(t)) e^{ip(t)x/\hbar} ,$$

where $x(t)$ and $p(t)$ are the solutions to classical SHO with initial condition $x(0) = x_0$ , $p(0) = 0$. Please interpret this result.

**Problem 1: The Coherent State (50 Points)**

Consider a wavepacket describing a mass $m$ in a harmonic potential of frequency $\omega$ consisting of a superposition of energy eigenstates defined as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n \, |n\rangle, \quad \text{where } c_n = e^{-|\alpha|^2/2} \frac{(\alpha)^n}{\sqrt{n!}}, \text{ with } \alpha \text{ a complex number.}$$

Such a state is know as a "coherent state" or sometime a "quasi-classical state.

(a) Show that $|\alpha\rangle$ is an eigenstate of the annihilation operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.
(b) Show that in this state $\langle \hat{x} \rangle = x_c \, \mathrm{Re}(\alpha)$, $\langle \hat{p} \rangle = p_c \, \mathrm{Im}(\alpha)$, where $x_c = \sqrt{2\hbar / m\omega}$, $p_c = \sqrt{2m\hbar\omega}$ .

One can show that, in position space, the wave function for this state is

$$\psi_\alpha(x) = e^{ip_0 x/\hbar} u_0(x - x_0)$$

where $u_0(x)$ is the ground state Gaussian wavepacket and $x_0 = x_c \, \mathrm{Re}(\alpha)$, $p_0 = p_c \, \mathrm{Im}(\alpha)$. This is just what was considered in the Extra Credit problem!

(c) What is the wave function in momentum space? Interpret $x_0$ and $p_0$.
(d) Explicitly show that $\psi_a(x)$ is an eigenstate of the annihilation operator using its position-space representation of the annihilation operator.
(e) Show that the coherent state is a minimum uncertainty wavepacket (with equal uncertainties in $x$ and $p$, in characteristic dimensionless units).
(f) If at time $t$=0 the state is $|\psi(0)\rangle = |\alpha\rangle$, show that at a later time,

$$|\psi(t)\rangle = e^{-i\omega t/2} \left| \alpha e^{-i\omega t} \right\rangle \text{ --interpret.}$$

(g) Show that, as a function of time, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ follow the *classical trajectory* of the harmonic oscillator, hence the name "quasiclassical state".
(h) Write the wave function as a function of time, $\psi_a(x,t)$. Sketch a cartoon of the time evolving probability density.

(i) Show that $\langle \hat{N} \rangle = |\alpha|^2$ and $\Delta N = |\alpha| = \sqrt{\langle N \rangle}$, so that in the classical limit $\lim_{|\alpha| \to \infty} \Delta N / \langle N \rangle \to 0$.
(j) The statistics associated with the number of excitations found in (i) are those of "counting statistics", i.e. a Poisson distribution, discussed last semester. Show that the probability distribution in $n$ is indeed Poissonian, with the appropriate parameters.

(k) Extra Credit **5 points**: The phase of a classical oscillator is $\phi(t) = \omega t - \phi_0$. Use the "rough" time-energy uncertainty principle $\Delta E \Delta t > \hbar$, to find an uncertainty principle between the number and phase of a quantum oscillator.