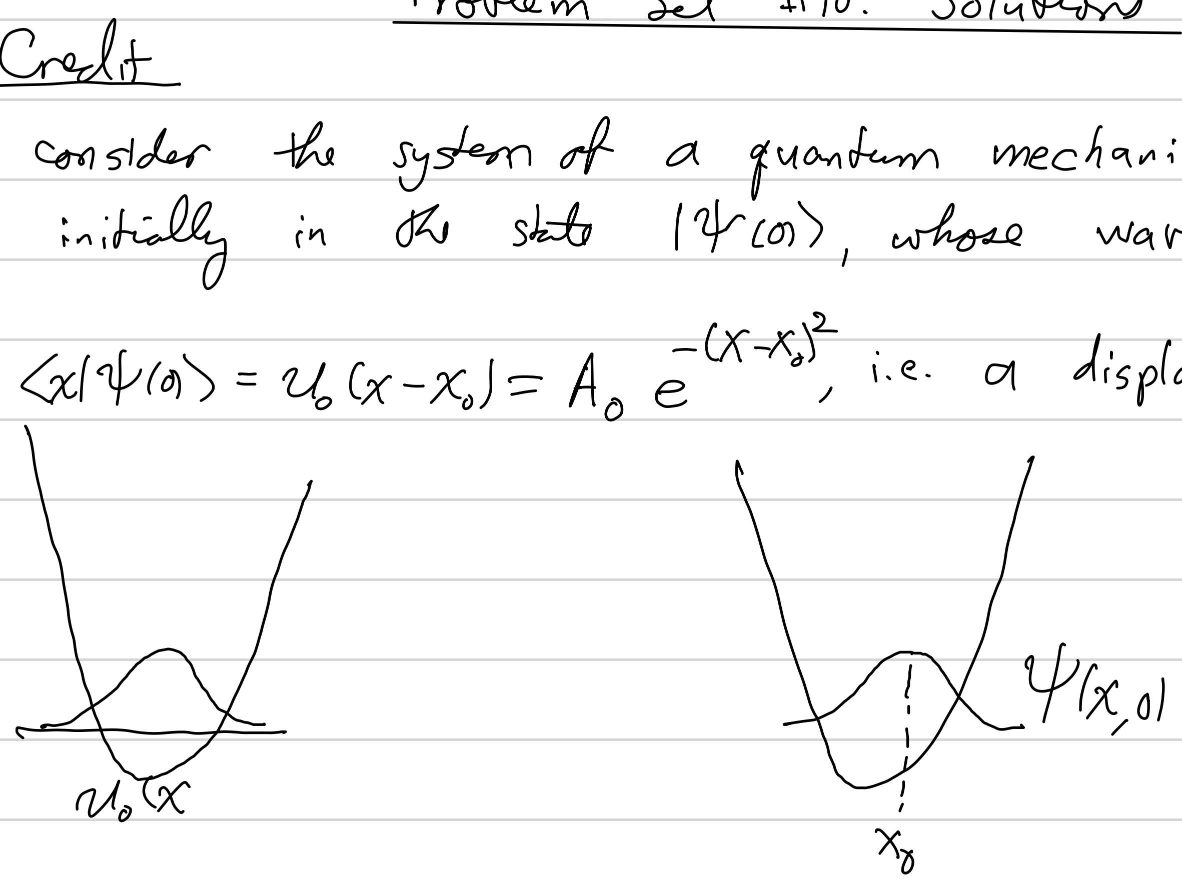
Physics 491: Quantum Mechanics I
Problem Set #10: Solutions

Extra Credit

(a) We consider the system of a quantum mechanical SHO, prepared initially in the state $|\psi(0)\rangle$, whose wave function is

$$\psi(x,0) = \langle x|\psi(0)\rangle = u_0(x-x_0) = A_0\, e^{-(X-X_0)^2}, \quad \text{i.e. a displaced ground state}$$



The initial state is not an eigenstate of the Hamiltonian. It can, however, be decomposed into a superposition of energy eigenstates, which form a basis for the Hilbert state

$$\psi(x,0) = \sum_{n=0}^{\infty} c_n u_n(x), \qquad u_n(x) = A_n \mathcal{H}(\sqrt{2}X)\, e^{-X^2}$$

$\mathcal{H}_n$ are Hermite polynomials, $X \equiv \frac{x}{x_c}$, $x_c = \sqrt{\frac{2\hbar}{m\omega}}$, $A_n = \frac{A_0}{\sqrt{n!\,2^n}}$, $A_0 = \frac{1}{(\pi \hbar/m\omega)^{1/4}}$

To find c_n we can use the "generating function" of the Hermite polynomial

$$\sum_{n=0}^{\infty} \mathcal{H}_n(z)\frac{y^n}{n!} = e^{2zy-y^2}$$

Let $z=\sqrt{2}X$, $y = \frac{X_0}{\sqrt{2}}$ $\Rightarrow$

$$\sum_{n=0}^{\infty} \mathcal{H}_n(\sqrt{2}X)\frac{(X_0/\sqrt{2})^n}{n!} = e^{2XX_0 - \frac{X_0^2}{2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} \underbrace{\frac{A_0}{\sqrt{n!\,2^n}}\mathcal{H}_n(\sqrt{2}X)\,e^{-X^2}}_{u_n(x)}\ \underbrace{\frac{X_0^n}{\sqrt{n!}}\,e^{-X_0^2/2}}_{c_n} = A_0\, e^{-X^2+2XX_0-X_0^2}$$

$$= A_0\, e^{-(X-X_0)^2} = u_0(x-x_0)$$

Thus, $\boxed{\psi(x,0) = \sum_n' c_n u_n(x) \quad \text{with} \quad c_n = \frac{e^{-\frac{X_0^2}{2}}}{\sqrt{n!}} X_0^n \, , \; X_0 = \frac{x_0}{x_c}}$

(b) At time $t=0$, the probability of finding the particle in the n^{th} excited state is

$$P_n = \langle n|\psi(0)\rangle = |c_n|^2 = e^{-X_0^2}\frac{(X_0^2)^n}{n!}$$

Note, this is a Poisson distribution, familiar in counting statistics

$$P_n = \frac{r^n}{n!}e^{-r},$$

$$\langle n\rangle = \sum_{n=0}^{\infty} n P_n = e^{-r}\sum_{n=0}^{\infty}\frac{n(r^n)}{n!} = e^{-r}\, r\sum_{n=1}^{\infty}\frac{r^{(n-1)}}{(n-1)!} = e^{-r}\, r \underbrace{\sum_{n=0}^{\infty}\frac{r^n}{n!}}_{e^{+r}} = r$$

Here $\langle\hat{N}\rangle = \sum n P_n = X_0^2 \Rightarrow \boxed{P_n = e^{-\langle\hat{N}\rangle}\frac{\langle\hat{N}\rangle^n}{n!}}$

(c) Because the initial state is not an eigenstate of the Hamiltonian, it is not a stationary state. Thus, it will evolve in time according to the Schrödinger equation. Given the decomposition into energy eigenstates,

$$\psi(x,t) = \sum_n' c_n e^{-iE_nt} u_n(x) = \sum_n c_n e^{-i(n+\frac{1}{2})\omega t} u_n(x)$$

$$= \cancel{e^{-\frac{i\omega t}{2}}} \sum_n e^{-X_0^2/2}\frac{(X_0 e^{-i\omega t})^n}{\sqrt{n!}} u_n(x) \quad \rightarrow \frac{A_0}{\sqrt{2^n n!}} H_n(\sqrt{2}X)e^{-X^2}$$

neglect overall constant phase

$$\Rightarrow \psi(x,t) = A_0 e^{-\frac{X_0^2}{2}} e^{-X^2} \underbrace{\sum_n' H_n(\sqrt{2}X)\frac{\left(\frac{X_0}{\sqrt{2}}e^{-i\omega t}\right)^n}{n!}}_{\exp\left(2X X_0 e^{-i\omega t} - \frac{X_0^2 e^{-2i\omega t}}{2}\right)}$$

$$\Rightarrow \psi(x,t) = A_0 \exp\left(-X^2 + 2X X_0 e^{-i\omega t} - X_0^2\left(\frac{1+e^{-2i\omega t}}{2}\right)\right)$$

$$\Rightarrow \psi(x,t) = A_0 \exp\left(-x^2 + 2x x_0 e^{-i\omega t} - x_0^2 e^{-i\omega t} \cos\omega t\right)$$

$$= A_0 \exp\left(-x^2 + 2x x_0 \cos\omega t - x_0^2 \cos^2\omega t\right) \exp\left(-2i x x_0 \sin\omega t\right) \exp\left(i x_0^2 \sin\omega t \cos\omega t\right)$$

$$\Rightarrow \psi(x) = A_0 \exp\left(-(x - X(t))^2\right) e^{2i x P(t)} \underbrace{e^{-2i X_0 P(t)}}_{\text{Phase factor indep. of } x}$$

where $X(t) = X_0 \cos\omega t$, $P(t) = -X_0 \sin\omega t$ is the solution to the classical SHO, with initial conditions $X(0) = X_0$, $P(0) = 0$. Putting back units

$$\boxed{\psi(x) = A_0 e^{-(x - x(t))^2} e^{\frac{i p(t) x}{\hbar}} = u_0(x - x(t)) e^{\frac{i p(t) x}{\hbar}}}$$

Thus, the wave packet sloshes back and forth as a Gaussian wave packet whose mean position oscillates as the classical SHO. The position dependent phase encodes the momentum as a function of time. This state is known as a <u>coherent state</u> or "quasi classical" state, because the wave packet oscillates as the classical SHO. We will study this again using operator methods.

Problem - Coherent State of the SHO

Definition $|\alpha\rangle \equiv \sum_{n=0}^{\infty} c_n |n\rangle$ (not stationary state)

where $c_n = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$

(a) $\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\, |n-1\rangle$

$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle$

(Aside: $c_{n+1}\sqrt{n+1} = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^{n+1}\sqrt{n+1}}{\sqrt{(n+1)!}} = e^{-|\alpha|^2/2} \frac{\alpha^{n+1}}{\sqrt{n!}}$

$= \alpha c_n$)

$\Rightarrow \hat{a}|\alpha\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle = \alpha |\alpha\rangle$

So the coherent state is an eigenstate of the annihilation operator.

(α Complex #)

(b) Recall $\hat{x} = x_c \hat{X} = x_c \left(\frac{\hat{a} + \hat{a}^\dagger}{2} \right)$ $\quad x_c = \sqrt{2\hbar/m\omega}$

$\hat{p} = p_c \hat{P} = p_c \left(\frac{\hat{a} - \hat{a}^\dagger}{2i} \right)$ $\quad p_c = \sqrt{2m\hbar\omega}$

Also $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^\dagger = \alpha^* \langle\alpha|$

$\Rightarrow \langle\alpha|\hat{a}|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$ $\quad$ ($|\alpha\rangle$ is normalized)

$\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^*\langle\alpha|\alpha\rangle = \alpha^*$

$$\Rightarrow \langle\alpha|\hat{x}|\alpha\rangle = x_c \left(\frac{\alpha + \alpha^*}{2} \right) = x_c \,\mathrm{Re}(\alpha)$$

$$\langle\alpha|\hat{p}|\alpha\rangle = p_c \left(\frac{\alpha - \alpha^*}{2i} \right) = p_c \,\mathrm{Im}(\alpha)$$

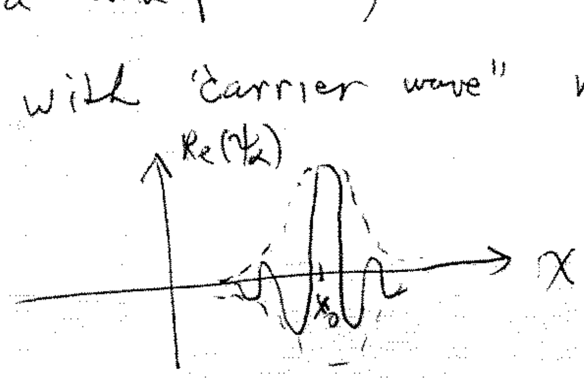
(c)

Given $\langle x|\alpha\rangle = e^{ip_0 x/\hbar}\, u_0(x - x_0) = \psi_\alpha(x)$

and $\alpha = \frac{x_0}{x_c} + i\frac{p_0}{x_c}$

We see that x_0 + p_0 are the mean position and momentum found in part (b).

$\psi_\alpha(x)$ is a wave packet, Gaussian, centered at $x = x_0$ with "carrier wave" momentum p_0



The momentum space wave function is the Fourier transform of $\psi_\alpha(x)$

$$\tilde{\psi}_\alpha(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} \psi_\alpha(x) \, e^{-ipx/\hbar}$$

We can use the convolution theorm

$$\tilde{\psi}_\alpha(p) = \underbrace{\tilde{\mathcal{F}}\left[e^{ip_0x/\hbar}\right]}_{\substack{\parallel \\ \delta(p-p_0)}} \circledast \tilde{\mathcal{F}}\left[u_0(x-x_0)\right]$$

Delta function $\rightarrow$ $\delta(p-p_0)$ $\underset{\uparrow \text{ convolution}}{\circledast}$ $\underbrace{e^{-ix_0p/\hbar} \, \tilde{u}_0(p)}_{\text{Shift theore}}$

$$\Rightarrow \boxed{\tilde{\psi}_\alpha(p) = \underset{\substack{\uparrow \\ \text{Overall} \\ \text{constant phase}}}{e^{\frac{-ix_0p_0}{\hbar}}} \left(e^{-ix_0p/\hbar} \, \tilde{u}_0(p-p_0) \right)}$$

In momentum space, $\tilde{u}_0$ centred at p_0.

the mean position appears as a phase in momentum space.

(d) We seek to show $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ in position space:

$$\langle x|\hat{a}|\alpha\rangle = \langle x|\hat{X}+i\hat{P}|\alpha\rangle = \left(\frac{x}{x_c} + \frac{i}{p_c}\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_\alpha(x)$$

$$= \left(\frac{x}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial x}\right) e^{ip_0 x/\hbar} u_0(x-x_0)$$

$$= \frac{x}{x_c} e^{ip_0 x/\hbar} u_0(x-x_0) + i\frac{p_0}{p_c} e^{ip_0 x/\hbar} u_0(x-x_0)$$

$$+ \frac{\hbar}{p_c} e^{ip_0 x/\hbar} \frac{\partial}{\partial x} u_0(x-x_0)$$

$$= e^{ip_0 x/\hbar} \underbrace{\left(\frac{x}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial x}\right) u_0(x-x_0)}_{\parallel} + i\frac{p_0}{p_c}\psi_\alpha(x)$$

let $y = x - x_0$

$$\Rightarrow \left(\frac{x_0}{x_c} + \underbrace{\frac{y}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial y}\right) u_0(y)}_{=0}$$

$\hat{a}\, u_0(y) = 0$

$$\therefore \langle x|\hat{a}|\alpha\rangle = \left(\frac{x_0}{x_c} + i\frac{p_0}{p_c}\right)\psi_\alpha(x)$$

$$= \alpha\, \psi_\alpha(x) \checkmark$$

(e) Uncertainties: $\Delta x = \sqrt{\Delta x^2}$, $\Delta p = \sqrt{\Delta p^2}$

$$\Delta x^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2, \quad \Delta p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

We have already found $\begin{cases} \langle \hat{x} \rangle = x_c \, \mathrm{Re}(\alpha) \\ \langle \hat{p} \rangle = p_c \, \mathrm{Im}(\alpha) \end{cases}$

$$\langle \hat{x}^2 \rangle = \frac{x_c^2}{4} \langle \alpha | \left(\hat{a}^2 + \hat{a}^{+2} + \hat{a}^+ \hat{a} + \hat{a}\hat{a}^+ \right) | \alpha \rangle$$

Use $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$, $\langle \alpha | \hat{a}^+ = \alpha^* \langle \alpha |$

$$\Rightarrow \langle \hat{x}^2 \rangle = \frac{x_c^2}{4} \Big(\alpha^2 + \alpha^{*2} + \alpha^* \alpha + \underbrace{\langle \alpha | \hat{a} \hat{a}^+ | \alpha \rangle}_{= \langle \alpha | \hat{a}^+ \hat{a} + 1 | \alpha \rangle = \alpha^* \alpha + 1} \Big)$$

$$\Rightarrow \langle \hat{x}^2 \rangle = \frac{x_c^2}{4} (\alpha + \alpha^*)^2 + \frac{x_c^2}{4}$$

$$= (x_c \, \mathrm{Re}(\alpha))^2 + \frac{x_c^2}{4}$$

$$\Rightarrow \Delta x^2 = \frac{x_c^2}{4} \Rightarrow \boxed{\Delta x = \frac{x_c}{2} = \sqrt{\frac{\hbar}{2m\omega}}}$$

Similarly $\Delta p^2 = \frac{p_c}{4} \Rightarrow \boxed{\Delta p = \frac{p_c}{2} = \sqrt{\frac{m\hbar\omega}{2}}}$

$\boxed{\Delta x \, \Delta p = \frac{\hbar}{2}}$ minimum uncertain wave packet

(f) at $t=0$ $\quad |\psi(0)\rangle = |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n \, \hat{U}(t)|n\rangle$$

$$= \sum_{n=0}^{\infty} c_n \, e^{-iE_nt/\hbar} |n\rangle \qquad \text{where } E_n = \hbar\omega(n+\tfrac{1}{2})$$

$$= e^{-i\omega t/2} \sum_{n=0}^{\infty} c_n e^{-in\omega t} |n\rangle$$

where $c_n = e^{-|\alpha|^2/2} \dfrac{\alpha^n}{\sqrt{n!}}$

$$\Rightarrow |\psi(t)\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} \sum_{n=0}^{\infty} e^{-|\alpha(t)|^2/2} \frac{(\alpha(t))^n}{\sqrt{n!}} |n\rangle$$

where $\alpha(t) = \alpha e^{-i\omega t}$

$$\Rightarrow \boxed{\begin{array}{c} |\psi(t)\rangle = e^{-i\omega t/2} |\alpha(t)\rangle \\ \text{where } \alpha(t) = \alpha e^{-i\omega t} \end{array}}$$

At every time, the state is a coherent state with eigenvalue that evolves in time as the classical complex amplitude.

(g) At later times

$$\langle \psi(t) | \hat{x} | \psi(t) \rangle = \langle \alpha(t) | \hat{x} | \alpha(t) \rangle = x_c \, \mathrm{Re}(\alpha(t))$$
$$= x_c \, \mathrm{Re}(\alpha e^{-i\omega t})$$

$$\langle \psi(t) | \hat{p} | \psi(t) \rangle = \langle \alpha(t) | \hat{p} | \alpha(t) \rangle = p_c \, \mathrm{Im}(\alpha(t))$$
$$= p_c \, \mathrm{Im}(\alpha e^{-i\omega t})$$

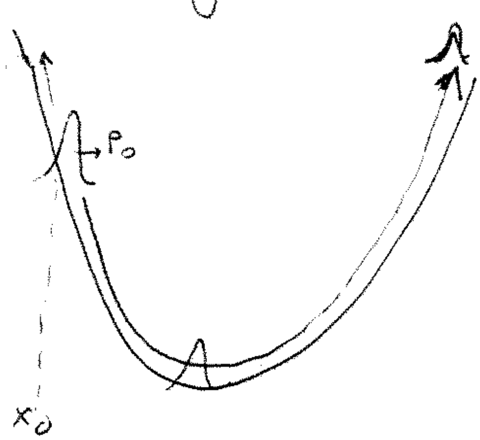
these are the classical equation of motion

(h) Wave functions:

$$\psi_{\alpha(t)}(x,t) = e^{i p(t) x / \hbar} \, u_0(x - x(t))$$

where $x(t)$ and $p(t)$ are the classical trajectories.

(i) Oscillating wave packet



Gaussian oscillating like a classical SHO

(i) $\langle \hat{N} \rangle = \langle \alpha | \hat{a}^{\dagger} \hat{a} | \alpha \rangle = \alpha^* \alpha = |\alpha|^2$

$$\Delta N = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}$$

$$\langle \hat{N}^2 \rangle = \langle \alpha | (\hat{a}^{\dagger} \hat{a})^2 | \alpha \rangle = \langle \alpha | \hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger} \hat{a} | \alpha \rangle$$

$$= \langle \alpha | \hat{a}^{\dagger 2} \hat{a}^2 | \alpha \rangle + \langle \alpha | \hat{a}^{\dagger} \underbrace{[\hat{a}, \hat{a}^{\dagger}]}_{= 1} \hat{a} | \alpha \rangle$$

$$= (\alpha^*)^2 (\alpha)^2 + (\alpha^*)(\alpha)$$

$$= |\alpha|^4 + |\alpha|^2$$

$$\Rightarrow \Delta N = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = \sqrt{|\alpha|^2} = |\alpha|$$

$$\Rightarrow \boxed{\Delta N = |\alpha|}$$

Thus $\boxed{\Delta N = \sqrt{\langle \hat{N} \rangle}}$

<u>Note:</u> $\lim_{|\alpha| \to \infty} \left(\frac{\Delta N}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}} = \frac{1}{|\alpha|} \right) = 0$

The <u>fractional</u> uncertainty goes to zero as the mean amplitude → <u>zero</u>

(j) The probability for find the particle in the n^{th} excited states is given by

$$P_n = |\langle n|\alpha\rangle|^2 = |C_n|^2 = \left| e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \right|^2$$

$$= e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!}$$

In part (i), we found $\langle n\rangle = |\alpha|^2$

$$\Rightarrow P_n = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}$$

This is none other than the <u>Poisson</u> distribution.

(k) The time-energy "uncertainty principle"

$\Delta E \Delta t \gtrsim \hbar$ (not quite a rigorous statement)

For the oscillator $E \sim n\hbar\omega$

$$\Rightarrow E t \sim n\hbar(\omega t)$$

$\uparrow$ phase of oscillator ϕ

$$\Rightarrow \boxed{\Delta n \Delta\phi \gtrsim 1}$$

This is the "number-phase" ~~as~~ uncertainty.

A quantum oscillator with <u>definite</u> n, has uncertain <u>phase</u>