University of New Mexico
Dept. of Physics and Astronomy

Physics 492 – Quantum Mechanics II

Final Exam

- Problems 1 and 2 **REQUIRED**.
- **Choose** one of Problem 3 **OR** Problem 4.
- Show all work – partial credit given.
- You may refer to any material distributed on the class web site and your notes.

Problem 1. Spin-Oscillator Coupling: (Required – 33 points)
Consider a Hamiltonian describing a spin-1/2 particle in a harmonic well given below,

$$\hat{H}_0 = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

(a) Show that $\left\{|n\rangle \otimes |\downarrow\rangle \equiv |n,\downarrow\rangle;\ |n\rangle \otimes |\uparrow\rangle \equiv |n,\uparrow\rangle\right\}$ are energy eigenstates with eigenvalues $E_{n,\downarrow} = n\hbar\omega$ and $E_{n,\uparrow} = (n+1)\hbar\omega$ respectively.

(b) The states associated with the ground energy and first excited energy level are,

$$\left\{|0,\downarrow\rangle, |1,\downarrow\rangle, |0,\uparrow\rangle\right\}$$

What is(are) the ground state(s)? What is(are) the first excited state(s)?
Note: Two states are degenerate.

(c) Now consider adding an interaction between the motion and the spin, described by Hamiltonian

$$\hat{H}_1 = \frac{\hbar\Omega}{2}\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right),$$

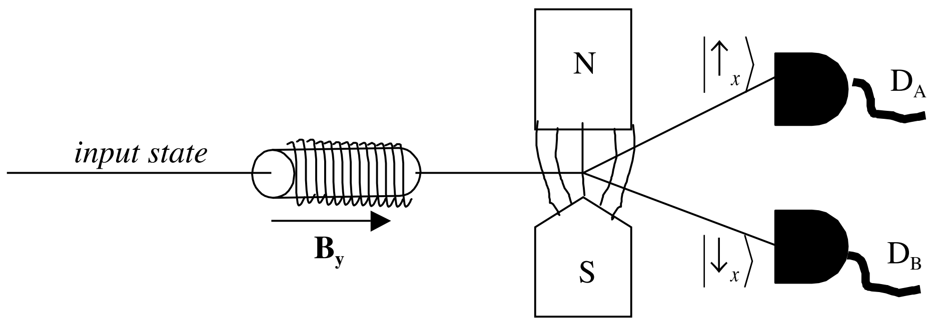
so that the total is now $\hat{H} = \hat{H}_0 + \hat{H}_1$.

Write a matrix representation of $\hat{H}$ in the subpace of ground and first excited states in the ordered basis given in part (b).

(d) Find the first order correction to ground and excited energy eigenvalues for the subspace above.

Problem 2. Measurement of a Spin-1/2 Particle (**Required 34 points**)

A spin-1/2 electron is sent through a solenoid with a uniform magnetic field in the *y*-direction and then measured with a Stern-Gerlach apparatus with gradient in the *x*-direction.



The time spent inside the solenoid is such that $\Omega t = \phi$, where $\Omega = \mu_B B_y / \hbar$ is the Larmor precession frequency.

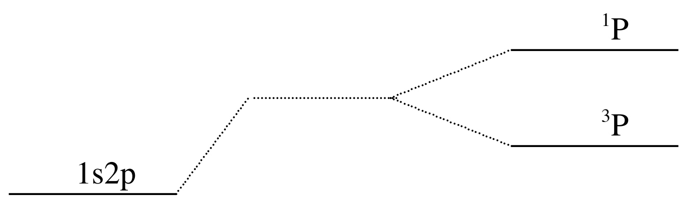
(a) Suppose the input state is the pure state $|\uparrow_z\rangle$. Show that the probability for detector D_A to fire as a function of ϕ is $P_{D_A} = \frac{1}{2}(\cos(\phi/2) + \sin(\phi/2))^2 = \frac{1}{2}(1+\sin\phi)$.
Repeat for the input state $|\downarrow_z\rangle$ and show $P_{D_A} = \frac{1}{2}(\cos(\phi/2) - \sin(\phi/2))^2 = \frac{1}{2}(1-\sin\phi)$.

(b) Now suppose the input is a pure coherent superposition of these states, $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$. Find and sketch the probability for detector D_A to fire as a function of ϕ.

(c) Now suppose the input state is the completely mixed state $\hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$. Find and sketch the probability for detector D_A to fire as a function of ϕ. Comment.

Problem 3: Helium (33 Points)

Consider a Helium atom in the $1s2p$ configuration. The total orbital angular momentum is L=1 (a P-state). Due to the Fermi symmetry this state splits into singlet and triplet multiplets shown below.



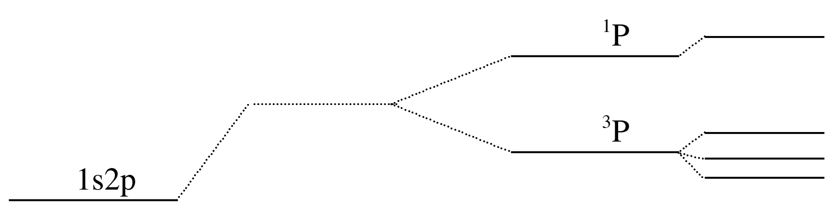
where the superscript 1 and 3 represent the spin degeneracy for singlet/triplet respectively.

(a) Explain qualitatively why the triplet state has lower energy.

Now include spin-orbit coupling described by the Hamiltonian, $\hat{H}_{so} = f(r)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$, where **L** and **S** are the total orbital and spin angular momentum respectively.

(b) Without the spin-orbit interaction, good quantum number for the angular momentum degrees of freedom are $|LM_L\ SM_S\rangle$. What are the good quantum number with spin-orbit?

(c) The energy level diagram including spin-orbit is sketched below. Label the states with appropriate quantum numbers



Note: Some of the levels are degenerate; the sublevels are not shown.

Problem 4. Hyperfine Interaction (33 points)

An important effect we neglected in our study of atomic spectra is the so-called "hyperfine interaction" -- the magnetic coupling between the electron spin and nuclear spin. Consider Hydrogen. The hyperfine interaction Hamiltonian has the form,

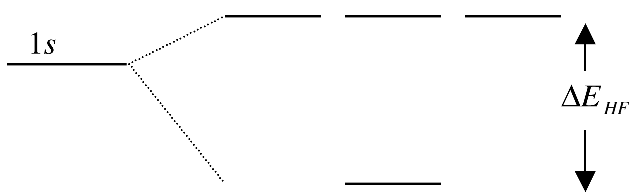
$$\hat{H}_{HF} = g_s g_i \mu_B \mu_N \frac{1}{r^3} \hat{\vec{s}} \cdot \hat{\vec{i}}$$

where $\hat{\vec{s}}$ is the electron's spin-1/2 angular momentum and $\hat{\vec{i}}$ is the proton's spin-1/2 angular momentum, and the appropriate g-factors and magnetons are given.

(a) In the absence of the hyperfine interaction, but including the electron and proton spin in the description, what is are the degeneracy of the ground state? Write all the quantum numbers associated with the degenerate sublevels.

(b) Now include the hyperfine interaction. Let $\vec{f} = \vec{i} + \vec{s}$ be the total spin angular momentum. Show that ground state manifold are defined with good quantum numbers $\left| n=1, l=0, s=1/2, i=1/2, f, m_f \right\rangle$. What are the possible values of f and m_f?

(c) The perturbed $1s$ ground state now has hyperfine splitting. The energy level diagram is sketched below. Label all the quantum numbers for the four sublevels shown.



(d) Show that the energy level splitting is $\Delta E_{HF} = g_s g_i \mu_B \mu_N \left\langle \frac{1}{r^3} \right\rangle_{1s}$. This splitting gives rises to the famous 21cm radio frequency radiation used in astrophysical observations.