Physics 492 - Quantum II

Final Exam Solutions

Problem 1: Spin-Oscillator Coupling

Given $\hat{H}_0 = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

(a) Set $\{|n\rangle \otimes |\uparrow\rangle, |n\rangle \otimes |\downarrow\rangle \mid n = 0, 1, \ldots\}$

$$\hat{H}_0 |n\rangle \otimes |\uparrow\rangle = \hbar\omega(\hat{N} + \tfrac{1}{2})|n\rangle \otimes |\uparrow\rangle + \frac{\hbar\omega}{2}|n\rangle \otimes \hat{\sigma}_z |\uparrow\rangle$$

$$= \left[\hbar\omega(n + \tfrac{1}{2}) + \frac{\hbar\omega}{2}\right]|n, \uparrow\rangle$$

$$\hat{H}_0 |n\rangle \otimes |\downarrow\rangle = \left[\hbar\omega(n + \tfrac{1}{2}) - \frac{\hbar\omega}{2}\right]|n, \downarrow\rangle$$

$\Rightarrow$ $|n, \uparrow\rangle$ and $|n, \downarrow\rangle$ are eigenstate of $\hat{H}_0$

(b) Eigenvalues: $E_{n,\uparrow} = \hbar\omega(n + 1)$

$E_{n,\downarrow} = \hbar\omega n$

Ground state: $|n=0, \downarrow\rangle$, $E_{0,\downarrow} = 0$

First excited state: $|n=1, \downarrow\rangle$; $|n=0, \uparrow\rangle$

$E_{1,\downarrow} = E_{0,\uparrow} = \hbar\omega$

doubly degenerate

(c) Add interaction $\hat{H}_1 = \frac{\hbar\Omega}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$

Matrix representation in ~~the~~ ground/first excited subspace

$$\{|n=0,\downarrow\rangle,\ |n=1,\downarrow\rangle;\ |n=0,\uparrow\rangle\}$$

$$\hat{H}_1|n=0,\downarrow\rangle = 0$$

$$\hat{H}_1|n=1,\downarrow\rangle = \frac{\hbar\Omega}{2}|n=0,\uparrow\rangle$$

$$\hat{H}_1|n=0,\uparrow\rangle = \frac{\hbar\Omega}{2}|n=1,\downarrow\rangle$$

$$\Rightarrow \hat{H}_1 \doteq \begin{array}{c} \begin{array}{ccc} |n=0,\downarrow\rangle & |n=1,\downarrow\rangle & |n=0,\uparrow\rangle \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\hbar\Omega}{2} \\ 0 & \frac{\hbar\Omega}{2} & 0 \end{bmatrix} \end{array}$$

(d) To find the first order corrections

- Ground State, non-degenerate

$$E^{(1)}_{0,\downarrow} = \langle 0,\downarrow|\hat{H}_1|0,\downarrow\rangle = 0$$

- First Excited State, degenerate

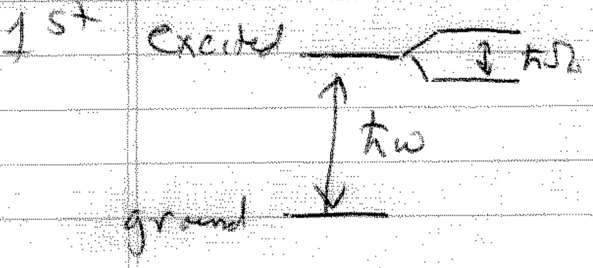
$\Rightarrow$ Find eigenvalues of $\hat{H}_1$ in sub space $\quad \hat{H}_1 \doteq \frac{\hbar\Omega}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Characteristic polynomial

$$\det[A_1 - \lambda \mathbb{1}] = \left(\frac{\hbar\Omega}{2}\right)^2 - \lambda^2 = 0$$

$$\Rightarrow \lambda = \pm \frac{\hbar\Omega}{2}$$

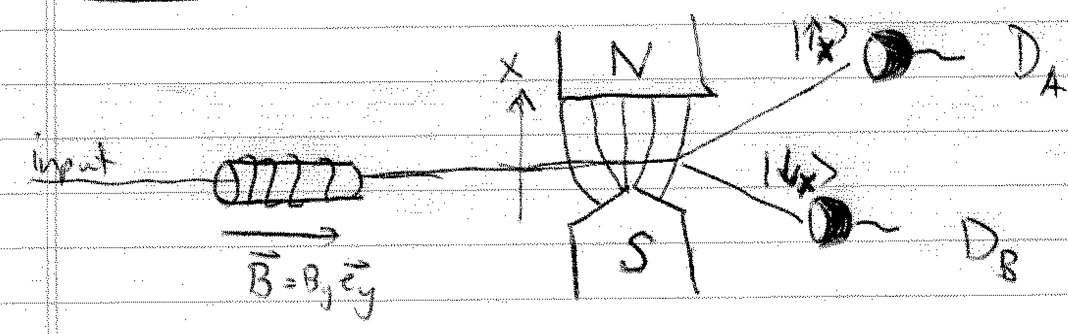
$\Rightarrow$ To first order $\quad E^{(1)} = \pm \frac{\hbar\Omega}{2}$



New eigenstates in first excited manifold

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,\uparrow\rangle \pm |1,\downarrow\rangle\right)$$

Problem 2: Measurement of a Spin-1/2 Particle



The effect of the solenoid is to rotate the spin according to the Hamiltonian $\hat{H} = \mu_B B_y \hat{\sigma}_y$

$$\hat{R}_y = e^{-i\hat{H}t/\hbar} = e^{-i\Omega t\hat{\sigma}_y} = e^{-i\phi\hat{\sigma}_y}$$

$$= \cos\left(\frac{\phi}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\phi}{2}\right)\hat{\sigma}_y$$

$$\doteq \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$ in standard basis

(a) Intial state $|\uparrow_z\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\hat{R}(\phi)|\uparrow_z\rangle \doteq \begin{bmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{bmatrix}$$

$$P_{D_A} = |\langle\uparrow_x|\hat{R}(\phi|\uparrow_z\rangle|^2 = \left| \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\phi/2 \\ \sin\phi/2 \end{bmatrix} \right|^2$$

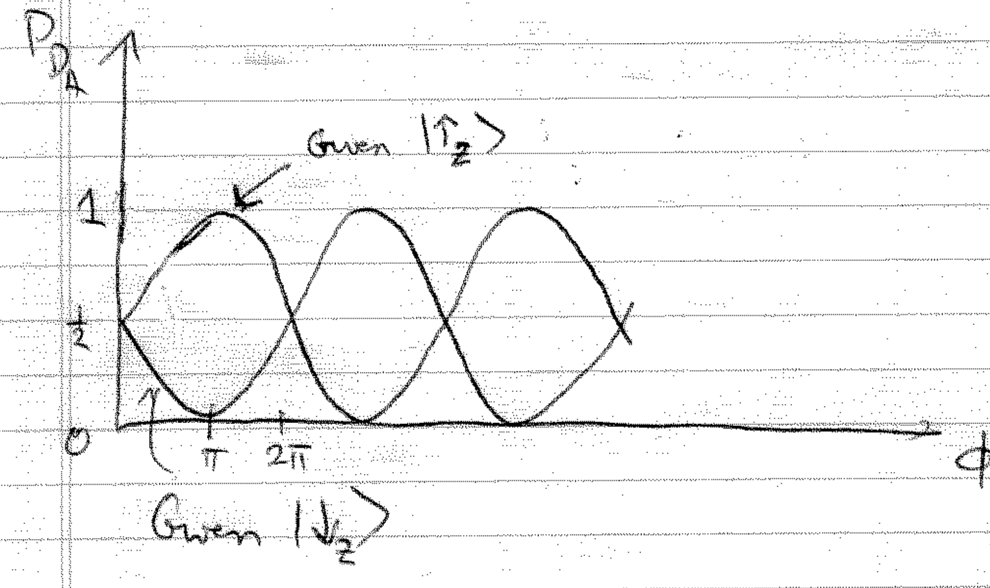
$$\boxed{P_{D_A} = \frac{\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\right)^2}{2} = \frac{1}{2}(1 + \sin\phi)}$$

- Initial state $|\downarrow_z\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\hat{R}(\phi)\,|\downarrow_z\rangle \doteq \begin{bmatrix} -\sin\phi/2 \\ \cos\phi/2 \end{bmatrix}$$

$$\Rightarrow P_{D_A} = \left|\langle\uparrow_x|\,\hat{R}(\phi)\,|\downarrow_z\rangle\right|^2 = \left| \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\sin\phi/2 \\ \cos\phi/2 \end{bmatrix} \right|^2$$

$$= \frac{1}{2}\left(-\sin\frac{\phi}{2} + \cos\frac{\phi}{2}\right)^2 = \frac{1}{2}\left(1 - \sin\phi\right)$$

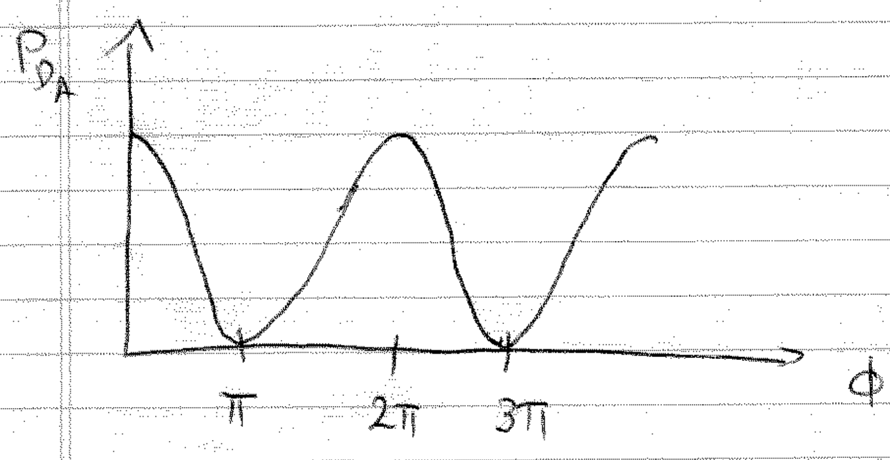


(b) Initial state $|\uparrow_x\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$\hat{R}(\phi)\,|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\\ \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\end{bmatrix}$$

$$P_{D_A} = \left|\langle\uparrow_x|\hat{R}(\phi)|\uparrow_x\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\frac{\cos\phi/2 - \sin\phi/2}{\sqrt{2}}\\ \frac{\cos\phi/2 + \sin\phi/2}{\sqrt{2}}\end{bmatrix}\right|^2$$

$$\Rightarrow \boxed{P_{D_A} = \left|\cos\frac{\phi}{2}\right|^2}$$



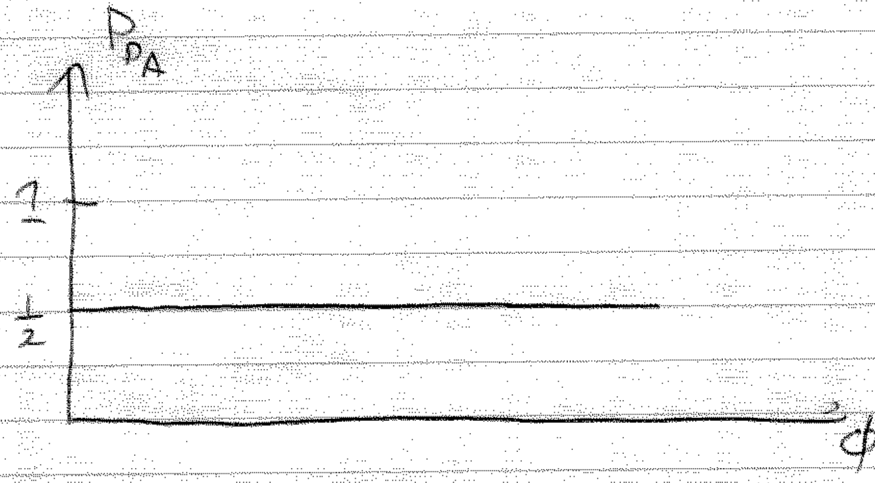
Still maximum contrast fringes

(c) Given Mixed state

$$\rho = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$$

$$P_{D_A} = \frac{1}{2}\underbrace{P(\uparrow_x|\uparrow_z)}_{\Downarrow\ \frac{1}{2}\left(1+\sin\frac{\phi}{2}\right)} + \frac{1}{2}\underbrace{P(\uparrow_x|\downarrow_z)}_{\Downarrow\ \frac{1}{2}\left(1-\sin\frac{\phi}{2}\right)}$$

$$\Rightarrow \boxed{P_{D_A} = \frac{1}{2} \quad \forall\phi}$$



No <u>interference</u> between two choices $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

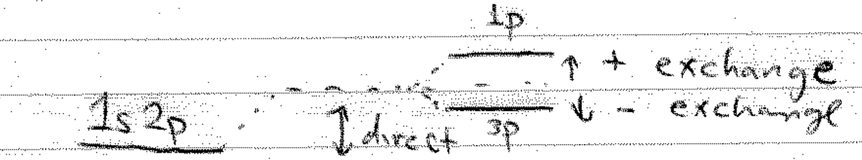
Problem 3: Helium

Given electron configuration $1s2p$

orbital $\ell_1 = 0$ $\ell_2 = 1$

Total orbital $L = 1 \Rightarrow P$ - state

Energy-level diagram:



The two-electron state

$$|\Psi\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|\phi_{1s}\rangle|\phi_{2p}\rangle \pm |\phi_{2p}\rangle|\phi_{1s}\rangle\right)}_{\text{Spatial wave function}} \otimes \underbrace{|\chi^{\mp}_{spin}\rangle}_{\text{Spin State}}$$

Fermi symmetry $\Rightarrow$
- symmetric space w/ antisymmetric spin
- antisymmetric space w/ sym spin

$$|\chi^{-}_{spin}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \quad \text{Singlet}$$

$$|\chi^{+}_{spin}\rangle = \left\{\begin{array}{c} |\uparrow\uparrow\rangle \\ \text{or} \\ |\downarrow\downarrow\rangle \\ \text{or} \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \end{array}\right\} \quad \text{triplet}$$

With symmetric spatial wave function electrons closer together ⇒ More repulsion

⇒ Higher energy

→ $\boxed{\text{Singlet Higher energy than Triplet}}$

(b) L-S Coupling $\hat{H}_{SO} = f(r)\, \hat{\vec{L}} \cdot \hat{\vec{S}}$

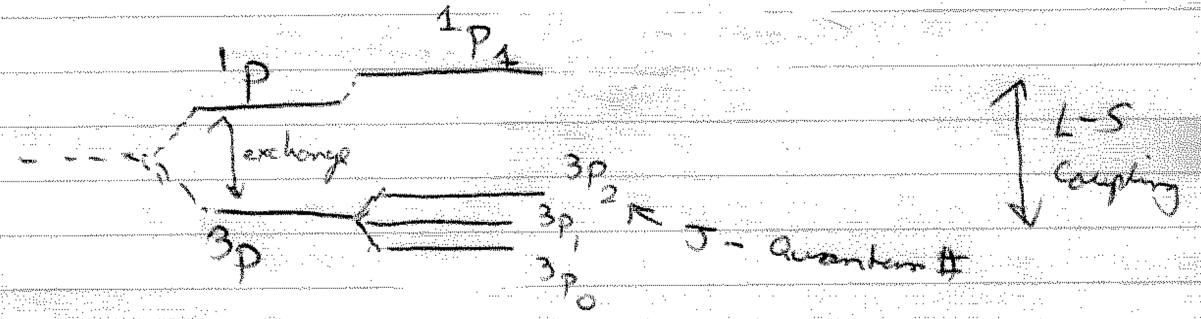
Good quantum numbers = coupled representation of $\vec{L}$ and $\vec{S}$,

$$|J \ M_J \ L \ S\rangle$$

(c) Possible J ~~val~~ values $|L-S| \le J \le L+S$

Singlet $S=0 \Rightarrow J = L = 1$

Triplet $S=1 \Rightarrow J = 2, 1, 0$



Problem 4: Hyperfine Interaction

Electron spin — Nuclear spin coupling

$$\hat{H}_1 = (g_s g_i \mu_B \mu_N) \frac{1}{r^3} \hat{\vec{s}} \cdot \hat{\vec{i}}$$

(a) In the absense of coupling between electron and proton spin the appropriate quantum numbers are the uncoupled rep.

The Ground State: 1s ← degeneracy 4 (up-down electron) (up-down proton)

Quantum #'s $|n=1, l=0, m_l=0, s=1/2, i=1/2, m_s, m_i\rangle$

$$m_s = \pm 1/2 \quad , \quad m_i = \pm 1/2$$

(b) Now include hyperfine interaction.

Let $\vec{f} = \vec{i} + \vec{s}$ (total spin angular momentum)

Note $\vec{f}^2 = \vec{s}^2 + \vec{i}^2 + 2\vec{s}\cdot\vec{i}$

$$\Rightarrow \vec{s}\cdot\vec{i} = \frac{1}{2}\left(\vec{f}^2 - \vec{s}^2 - \vec{i}^2\right)$$

$\Rightarrow$ Eigenstates of $\hat{H}_{HF}$ are coupled $\vec{s}$ and $\vec{i}$
$\{\hat{f}^2, \hat{s}^2, \hat{i}^2, \hat{f}_z\}$

$\Rightarrow$ Good quantum #'s of ground state beyond 1s : $|n=1, l=0, s=1/2, i=1/2, f, m_f\rangle$

The q-number f ranges $|s-i| \le f \le s+i$

With $s = i = 1/2 \Rightarrow$ $\boxed{\begin{array}{l} f=0 \Rightarrow m_f = 0 \\ \text{or} \quad f=1 \Rightarrow m_f = 1, 0, -1 \end{array}}$

(c) using $\vec{F}^2 |f m_f s i\rangle = f(f+1)|f m_f s i\rangle$

$$\hat{S}^2 |f m_f s i\rangle = s(s+1)|f m_f s i\rangle$$
$$= \frac{3}{4}|f m_f s i\rangle$$
$$\hat{i}^2 |f m_f s i\rangle = i(i+1)|f m_f s i\rangle$$
$$= \frac{3}{4}|f m_f s i\rangle$$

$$\Rightarrow \hat{S}\cdot\hat{i}\,|f m_f s i\rangle = \frac{1}{2}\left[f(f+1) - \frac{3}{2}\right]|f m_f s i\rangle$$

$f=0$: $\hat{S}\cdot\hat{i}\,|0 0 \tfrac{1}{2} \tfrac{1}{2}\rangle = -\frac{3}{4}|0 0 \tfrac{1}{2} \tfrac{1}{2}\rangle$

$f=1$ $\vec{S}\cdot\vec{i}\,|1 m_f \tfrac{1}{2} \tfrac{1}{2}\rangle = +\frac{1}{4}|1 m_f \tfrac{1}{2} \tfrac{1}{2}\rangle$

$$\Rightarrow E_{1s\,(f=0)} = -\frac{3}{4}\, g_e g_p \mu_B \mu_N \left\langle \frac{1}{r^3} \right\rangle_{1s}$$
$$E_{1s\,(f=1)} = +\frac{1}{4}\, g_e g_p \mu_B \mu_N \left\langle \frac{1}{r^3} \right\rangle_{1s}$$

$$\Rightarrow \boxed{\Delta E_{HF} = E_{1s\,(f=1)} - E_{1s\,(f=0)} = g_e g_p \left\langle \frac{1}{r^3} \right\rangle_{1s}}$$

Level diagram: