# Physics 492 - Quantum II

# Midterm Exam Solutions

**Problem 1:** A wave packet of a SHO

At $t=0$ $\quad |\psi(0)\rangle = \sqrt{\frac{2}{3}}\,|0\rangle + \frac{i}{\sqrt{3}}|2\rangle$

(a) Average number of excitation $\langle \hat{N} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle$.
Since the energy eigenstates are number states

$$\langle \hat{N} \rangle = \sum_n P_n n = 0P_0 + 2P_2 = 2\left|\frac{i}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

The uncertainty $\Delta N = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}$

$$\langle \hat{N}^2 \rangle = \sum_n P_n n^2 = 4P_2 = 4\left|\frac{i}{\sqrt{3}}\right|^2 = \frac{4}{3}$$

$$\Rightarrow \Delta N = \sqrt{\frac{4}{3} - \left(\frac{2}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3}$$

(b) Since $[\hat{N}, \hat{H}] = 0$, the probability distribution $P_n$ is conserved $\Rightarrow$ $\langle \hat{N} \rangle$ and $\Delta N$ remain unchanged with time.

(c) Position $\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ in characteristic unit $x_c = \sqrt{\frac{2\hbar}{m\omega}}$

Momentum $\hat{P} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$ " " " $P_c = \sqrt{2m\hbar\omega}$

$$\Rightarrow \langle \hat{X} \rangle = \frac{1}{2}\left(\langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle\right) = \mathrm{Re}(\langle \hat{a} \rangle)$$

$$\langle \hat{P} \rangle = \frac{1}{2i}\left(\langle \hat{a} \rangle - \langle \hat{a}^\dagger \rangle\right) = \mathrm{Im}(\langle \hat{a}^\dagger \rangle)$$

As a function of time $|\psi(t)\rangle = c_0(t)|0\rangle + c_2(t)|2\rangle$

where $c_0(t) = \sqrt{\frac{2}{3}}\, e^{-iE_0t/\hbar} = \sqrt{\frac{2}{3}}\, e^{-i\omega t/2}$

$$c_2(t) = \frac{i}{\sqrt{3}}\, e^{-iE_2t/\hbar} = \frac{i}{\sqrt{3}}\, e^{-i5\omega t/2}$$

Aside: $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \Rightarrow \langle n'|\hat{a}|n\rangle = \delta_{n',n-1}\sqrt{n}$

$\Rightarrow$ $\hat{a}$ has no matrix elements between $|0\rangle$ and $|2\rangle$

$\Rightarrow \langle\hat{a}\rangle = 0 \ \forall$ times $\Rightarrow \boxed{\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0 \text{ for all } t}$

Aside 2: We could see this by parity since $|0\rangle$ and $|2\rangle$ are both even parity $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$

(d) Uncertainties $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \sqrt{\langle\hat{x}^2\rangle}$

$$\Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \sqrt{\langle\hat{p}^2\rangle}$$

Dimensionless units $\langle\hat{X}^2\rangle = \left\langle\left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)^2\right\rangle = \frac{1}{4}\langle\hat{a}^2 + \hat{a}^{\dagger 2} + \underbrace{\hat{a}^\dagger\hat{a}}_{\hat{N}} + \underbrace{\hat{a}\hat{a}^\dagger}_{\hat{N}+1}$

$\Rightarrow \langle\hat{X}^2\rangle = \frac{1}{2}\mathrm{Re}(\langle\hat{a}^2\rangle) + \frac{\langle\hat{N}\rangle}{2} + \frac{1}{4}$

Now $\langle\hat{a}^2\rangle = c_0^*(t)c_2(t)\langle 0|\hat{a}^2|2\rangle = \left(\sqrt{\frac{2}{3}}e^{+i\frac{\omega t}{2}}\right)\left(\frac{i}{\sqrt{3}}e^{-i\frac{5\omega t}{2}}\right)\sqrt{2} = \frac{2}{3}ie^{-i2\omega_0 t}$

$\Rightarrow \mathrm{Re}(\langle\hat{a}^2\rangle) = \frac{2}{3}\sin(2\omega_0 t)$

$\Rightarrow \langle\hat{X}^2\rangle = \frac{1}{3}\sin(2\omega t) + \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{4} = \frac{1}{3}\sin(2\omega t) + \frac{7}{12}$

Similarly $\langle \hat{P}^2 \rangle = \left\langle \left( \frac{\hat{a} - \hat{a}^\dagger}{2i} \right)^2 \right\rangle = -\frac{1}{4} \langle \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger \rangle$

$$= -\frac{1}{2} \mathrm{Re}\left( \langle \hat{a}^2 \rangle \right) + \frac{\langle \hat{N} \rangle}{2} + \frac{1}{4}$$

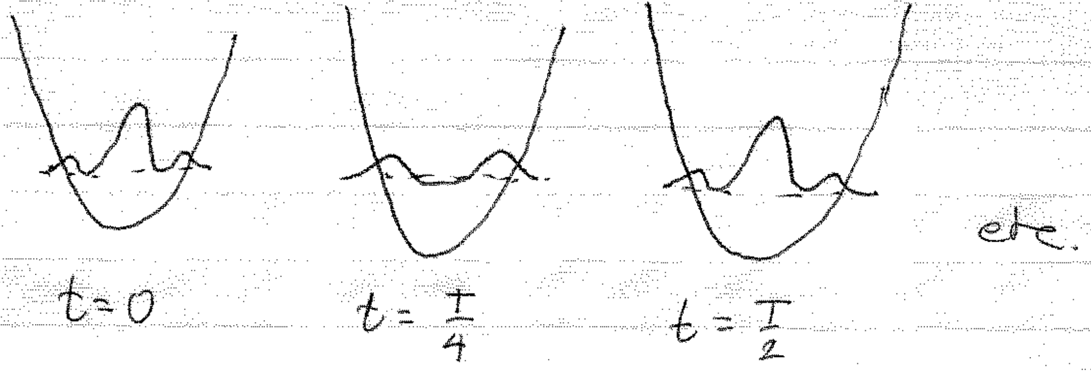
$$\Rightarrow \langle \hat{P}^2 \rangle = -\frac{1}{3} \sin(2\omega t) + \frac{7}{12}$$

$$\Rightarrow \boxed{\begin{array}{ll} \Delta x(t) = x_c \sqrt{\frac{7}{12} + \frac{1}{3} \sin(2\omega t)} & x_c = \sqrt{\frac{2m\omega}{\hbar}} \\ \Delta p(t) = p_c \sqrt{\frac{7}{12} - \frac{1}{3} \sin(2\omega t)} & p_c = \sqrt{2m\hbar\omega} \end{array}}$$

This is not a minimum uncertainty wave packet since $\Delta x \, \Delta p > \frac{\hbar}{2}$ for all times

(e) This wave packet "breathes", i.e. its width periodically oscillates, but it mean position stays fixed.

$$|\psi(x,t)|^2 = \frac{1}{3} \left| u_0(x) + i\sqrt{2}\, e^{-i2\omega_0 t} u_2(x) \right|^2$$



$$T = \frac{2\pi}{\omega}$$

## Problem 2: Spin $^3/_2$

Standard basis $|j,m\rangle$ : $\begin{cases} \hat{J}^2|j,m\rangle = j(j+1)\,|j,m\rangle \\ \hat{J}_z\,|j,m\rangle = \quad m\,|j,m\rangle \end{cases}$

$-j \le m \le j$ $\quad (2j+1$ values$)$

Here $j = ^3/_2 \Rightarrow$ 4 sublevels, abreviate $|m\rangle$

Given $\hat{J}_+ = \sqrt{3}\,|\tfrac{3}{2}\rangle\langle\tfrac{1}{2}| + 2\,|\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}| + \sqrt{3}\,|-\tfrac{1}{2}\rangle\langle-\tfrac{3}{2}|$

(a) $\hat{J}_- = \hat{J}_+^{\dagger} = \sqrt{3}\,|\tfrac{1}{2}\rangle\langle\tfrac{3}{2}| + 2\,|-\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| + \sqrt{3}\,|-\tfrac{3}{2}\rangle\langle-\tfrac{1}{2}|$

(b) Matrix representation in $\{|\tfrac{3}{2}\rangle, |\tfrac{1}{2}\rangle, |-\tfrac{1}{2}\rangle, |-\tfrac{3}{2}\rangle\}$

$\hat{J}_z$ is diagonal : $\quad \hat{J}_z \doteq \begin{bmatrix} +\frac{3}{2} & & 0 & \\ & +\frac{1}{2} & & \\ & 0 & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{bmatrix}$

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \qquad \hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

$$\hat{J}_+ \doteq \begin{array}{c} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ |\tfrac{3}{2}\rangle\ |\tfrac{1}{2}\rangle\ |-\tfrac{1}{2}\rangle\ |-\tfrac{3}{2}\rangle \end{array} \qquad \hat{J}_- \doteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\Rightarrow \hat{J}_x \doteq \frac{1}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \qquad \hat{J}_y \doteq \frac{1}{2i}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

(c) Given $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\,|\tfrac{3}{2}\rangle + |\tfrac{1}{2}\rangle - |-\tfrac{1}{2}\rangle - \sqrt{3}\,|-\tfrac{3}{2}\rangle\right)$

$$\doteq \frac{1}{2\sqrt{2}}\begin{bmatrix}\sqrt{3}\\ 1\\ -1\\ -\sqrt{3}\end{bmatrix}$$ in standard basis

Normalization: $\langle\psi|\psi\rangle = \frac{1}{8}\left(3+1+1+3\right) = 1$ ✓

$$\hat{J}_x|\psi\rangle \doteq \frac{1}{2}\begin{bmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix}\sqrt{3}\\ 1\\ -1\\ -\sqrt{3}\end{bmatrix} = \frac{1}{4\sqrt{2}}\begin{bmatrix}\sqrt{3}\\ 1\\ -1\\ -\sqrt{3}\end{bmatrix} = \frac{1}{2}|\psi\rangle$$

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$$\Rightarrow \boxed{|\psi\rangle \text{ is an eigenvector of } \hat{J}_x \text{ with eigenvalue } \tfrac{1}{2}}$$

Note: The problem had a typo and asked for the state with eigenvalue $m_x = 3/2$

That state is $|\psi_{3/2}\rangle = \frac{1}{2\sqrt{2}}\left(|\tfrac{3}{2}\rangle + \sqrt{3}\,|\tfrac{1}{2}\rangle + \sqrt{3}\,|-\tfrac{1}{2}\rangle + |-\tfrac{3}{2}\rangle\right)$

$$\hat{J}_x|\psi_{3/2}\rangle = \frac{1}{2}\begin{bmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{bmatrix} = \frac{1}{4\sqrt{2}}\begin{bmatrix}3\\ 3\sqrt{3}\\ 3\sqrt{3}\\ 3\end{bmatrix} = \frac{3}{2}|\psi_{3/2}\rangle$$

✓

(d) We suppose at $t=0$ we state in $|\psi_{3/2}\rangle$, describing a positive nucleus with g-factor $g_N$

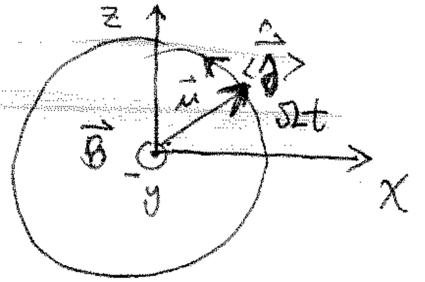
$$\Rightarrow \quad \hat{\mu} = g_N \frac{\mu_N}{\hbar} \hat{J} \qquad \text{where } \mu_N = \frac{e\hbar}{2Mc}$$

= nuclear magneton

A magnetic field is suddenly turned on in y-direction

$\Rightarrow$ Larmor precession about y-axis at frequency

$$\boxed{\Omega = g_N \mu_N B_y}$$



The angular momentum exprences a torque about $-\vec{B}$

$$\Rightarrow \boxed{\begin{aligned} \langle\hat{J}_x\rangle &= \frac{3}{2}\cos\Omega t \\ \langle\hat{J}_z\rangle &= \frac{3}{2}\sin\Omega t \\ \langle\hat{J}_y\rangle &= 0 \end{aligned}}$$