Physics 492 - Quantum II

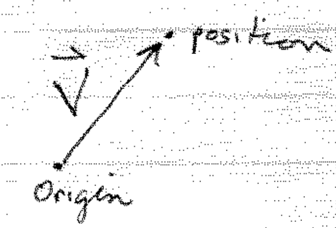
Lecture 2: Review Linear Vector Spaces

We have seen that the basic structure of quantum theory rests on the foundation of Hilbert space. In this lecture we firm up that foundation, reviewing the mathematics of linear vector spaces

Vector Spaces:

Vectors as abstract objects $\{\vec{V}\}$

E.g. position on plane
(Cartesian space $\mathbb{R}^2$)



- Vector space $\mathcal{H}$

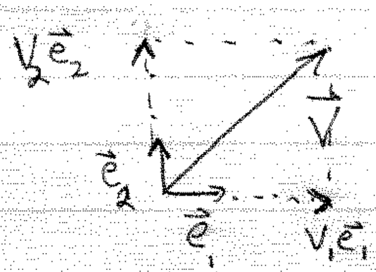
Vectors and "scalars" $\{\alpha\}$

Linear $\alpha \vec{V} + \beta \vec{U} = \vec{W} \in \mathcal{H}$

Basis: Linearly indepent set which spans the space

$$\{\vec{e}_i \mid i = 1, 2, \ldots, d\}$$ ← dimension of $\mathcal{H}$

$$\vec{V} = \sum_{i=1}^{d} V_i \, \vec{e}_i$$



Representation: Column vector

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \doteq \vec{V}$$

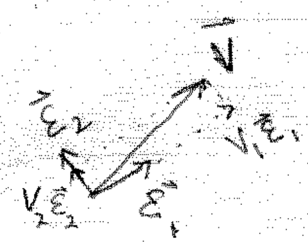
↑ representation in basis

Different representations possible

Another basis $\{\vec{\varepsilon}_i \mid i = 1, 2, \ldots, d\}$

$$\vec{V} = \sum_{i=1}^{d} \tilde{V}_i \, \vec{\varepsilon}_i$$

$$\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} \doteq \vec{V}$$



$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}$ are related by a change of basis (more on this later)

<u>Inner product</u>: We can endow our vector space with another structure, the "inner product". This gives a measure of <u>distance</u> and length

In Cartesian space this is the familiar "dot"

$$\vec{V} \cdot \vec{W} = |\vec{V}| \, |\vec{W}| \cos\theta$$

(projection)

"Norm" $|\vec{V}| = \sqrt{\vec{V} \cdot \vec{V}}$

Orthogonal vectors $\vec{V} \cdot \vec{W} = 0$

Unit vectors $|\vec{V}| = 1 \Rightarrow \vec{V} \cdot \vec{V} = 1$

Orthonormal basis $\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

The inner product is "linear"

$$\vec{V}\cdot(\alpha\vec{U}+\beta\vec{W}) = \alpha(\vec{V}\cdot\vec{U})+\beta(\vec{V}\cdot\vec{W})$$

$\Rightarrow$ In an orthonormal basis, the representation is easily calculated:

$$\vec{V}=\sum_i V_i\vec{e}_i \Rightarrow \vec{e}_j\cdot\vec{V}=\sum_i V_i\,\vec{e}_j\cdot\vec{e}_i = \sum_i V_i\,\delta_{ij}$$

$$\Rightarrow \boxed{\vec{e}_j\cdot\vec{V} = V_j}$$

Sometimes it is convenient to perform operations using the representation in a basis

$$\vec{V}\cdot\vec{W} = \left(\sum_i V_i\vec{e}_i\right)\cdot\left(\sum_j W_j\vec{e}_j\right) = \sum_{ij} V_i W_j \underbrace{(\vec{e}_i\cdot\vec{e}_j)}_{\delta_{ij}}$$

$$\Rightarrow \vec{V}\cdot\vec{W} = \sum_i V_i W_i$$

$$|\vec{V}|^2 = \sum_i V_i^2 \qquad \text{(Pythagoras theorem)}$$

Matrix algebra rules

$$[V_1\ V_2 \dots V_d]\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_d \end{bmatrix} = V_1W_1 + V_2W_2 + \dots V_dW_d$$

$$= \sum_i V_i W_j = \vec{V}\cdot\vec{W}$$

Complex vector space: e.g. $\mathbb{C}^d$ $\quad$ $\mathbb{C}$ = complex #

Two dimension vector space with complex scalars

Want $\mathbb{R}^d$ to be a subspace

Consider $d=2$: Basis $\vec{e}_+ = \dfrac{\vec{e}_1 + i\vec{e}_2}{\sqrt{2}}$ $\quad$ $\vec{e}_- = \dfrac{\vec{e}_1 - i\vec{e}_2}{\sqrt{2}}$

Where $\{\vec{e}_1, \vec{e}_2\}$ orthonormal basis for $\mathbb{R}^2$

$$\vec{e}_1 = \frac{\vec{e}_+ + \vec{e}_-}{\sqrt{2}}, \qquad \vec{e}_2 = \frac{\vec{e}_+ - \vec{e}_-}{i\sqrt{2}}$$

$$\vec{V} = V_1 \vec{e}_1 + V_2 \vec{e}_2 = \left(\frac{V_1 - iV_2}{\sqrt{2}}\right)\vec{e}_+ + \left(\frac{V_1 + iV_2}{\sqrt{2}}\right)\vec{e}_-$$

$$= V_+ \vec{e}_+ + V_- \vec{e}_- \quad \text{where } V_\pm = \frac{V_1 \mp iV_2}{\sqrt{2}}$$

Norm squared: $|\vec{V}|^2 = V_1^2 + V_2^2 = |V_+|^2 + |V_-|^2$

$\Rightarrow$ In $\mathbb{C}^d$ we can express the norm

$$|\vec{V}|^2 = \begin{bmatrix} V_+^* & V_-^* \end{bmatrix} \begin{bmatrix} V_+ \\ V_- \end{bmatrix} = V_+^* V_+ + V_-^* V_-$$

Matrix operation: Adjoint (dagger $\dagger$)

$$\begin{bmatrix} V_+ \\ V_- \end{bmatrix}^\dagger = \begin{bmatrix} V_+^* \\ V_-^* \end{bmatrix}^T = \begin{bmatrix} V_+^* & V_-^* \end{bmatrix}$$

transpose
row ⇄ column

We can define the adjoint abstractly as operation on abstract vector $\vec{V}$

$$\vec{V} \text{ vector} \Rightarrow \vec{V}^{\dagger} = \text{"dual vector"}$$

s.t. $\vec{V}^{\dagger} \cdot \vec{V} = |\vec{V}|^2$

Using the dot-product from $\mathbb{R}^d$, the adjoint in $\mathbb{C}^d$ must obey

$$(\alpha \vec{V} + \beta \vec{W})^{\dagger} = \alpha^* \vec{V}^{\dagger} + \beta^* \vec{W}^{\dagger}$$

Thus: $\vec{e}_+^{\dagger} = \dfrac{\vec{e}_x^{\dagger} - i\,\vec{e}_y^{\dagger}}{\sqrt{2}}$ $\qquad \vec{e}_-^{\dagger} = \dfrac{\vec{e}_x^{\dagger} + i\,\vec{e}_y^{\dagger}}{\sqrt{2}}$

$$\Rightarrow \vec{e}_+^{\dagger} \cdot \vec{e}_+ = \left(\frac{\vec{e}_x^{\dagger} - i\vec{e}_y^{\dagger}}{\sqrt{2}}\right) \cdot \left(\frac{\vec{e}_x + i\vec{e}_y}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\underbrace{(\vec{e}_x^{\dagger} \cdot \vec{e}_x)}_{1} + \frac{(-i)(i)}{2}\underbrace{(\vec{e}_y^{\dagger} \cdot \vec{e}_y)}_{1}$$

$$+ \frac{i}{2}\underbrace{(\vec{e}_x^{\dagger} \cdot \vec{e}_y - \vec{e}_y^{\dagger}\,\vec{e}_x)}_{0}$$

$\vec{e}_+^{\dagger} \cdot \vec{e}_+ = \underline{1}$ $\qquad$ Similarly $\vec{e}_-^{\dagger} \cdot \vec{e}_- = \underline{1}$

$$\vec{e}_+^{\dagger} \cdot \vec{e}_- = 0 = \vec{e}_-^{\dagger} \cdot \vec{e}_+$$

$\Rightarrow \{\vec{e}_+, \vec{e}_-\}$ orthonormal w.r.t. inner product

Note: [illegible]

Note: In Cartesian space we typically make no distinction between $\vec{e}_i$ and $\vec{e}_i^{\,\dagger}$ (column vs. row vector)

With respect to this inner product, in complex vector space

$$\left(\vec{V}^\dagger \cdot \vec{W}\right)^* = \vec{W}^\dagger \cdot \vec{V}$$

Operators (Linear):

Map on space $\hat{A} : \mathcal{H} \longrightarrow \mathcal{H}$

$$\hat{A}\,\vec{V} = \vec{W}$$

Linear $\quad \hat{A}(\alpha\vec{V} + \beta\vec{U}) = \alpha(\hat{A}\vec{V}) + \beta(\hat{A}\vec{U})$

Example rotation

If we know action of $\hat{A}$ on basis the we know action on any vector

$$\hat{A}\,\vec{V} = \sum_i V_i\left(\hat{A}\vec{e}_i\right) \equiv \vec{W}$$

Rotation example:

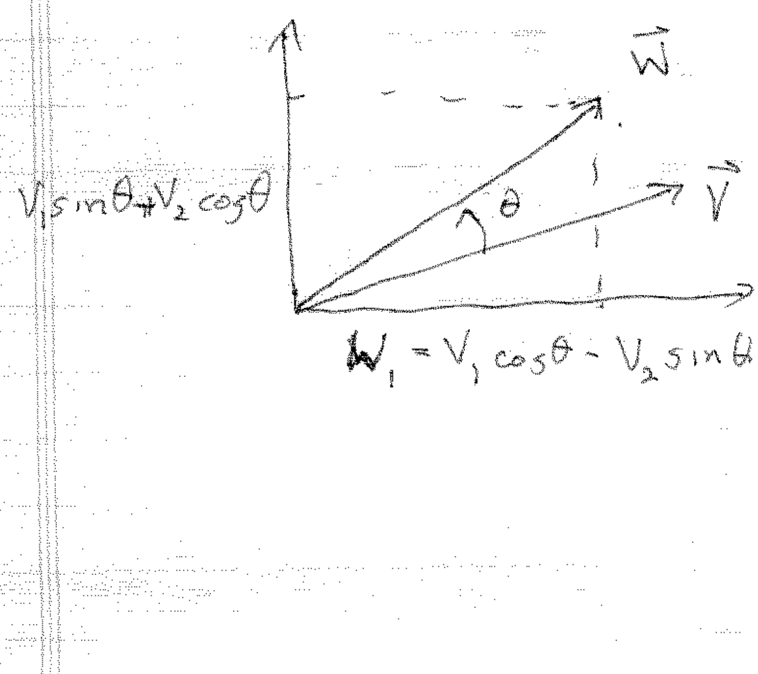
$$A_{11} = \vec{e}_1^{\,\dagger} \cdot (\hat{A}\vec{e}_1) = \cos\theta \qquad A_{12} = \vec{e}_1^{\,\dagger} \cdot \hat{A}\vec{e}_2 = -\sin\theta$$

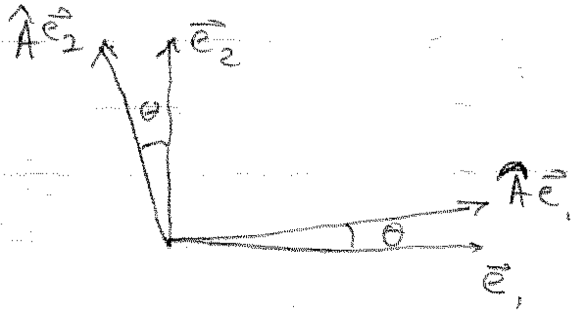
$$A_{21} = \vec{e}_2^{\,\dagger} \cdot (\hat{A}\vec{e}_1) = \sin\theta \qquad A_{22} = \vec{e}_2^{\,\dagger} \cdot \hat{A}\vec{e}_1 = \cos\theta$$

$$\Rightarrow \quad \hat{A} \doteq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= \begin{bmatrix} V_1\cos\theta - V_2\sin\theta \\ V_1\sin\theta + V_2\cos\theta \end{bmatrix}$$





$$\hat{A}\vec{e}_1 = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2 \,, \qquad \hat{A}\vec{e}_2 = -\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2$$

$$W_j = \vec{e}_j^{\,\dagger}\cdot\vec{W} = \sum_i V_i \underbrace{\left(\vec{e}_j^{\,\dagger}\cdot\hat{A}\vec{e}_i\right)}_{\equiv A_{ji} \text{ (number)}}$$

$$\Rightarrow \quad W_j = \sum_i A_{ji} V_i$$

Column vector

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_d \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{21} & A_{22} & & \\ \vdots & & & \\ A_{d1} & \cdot & - \cdots & A_{dd} \end{bmatrix}}_{\substack{\text{matrix} \\ \text{representation of } \hat{A} \\ \text{w.r.t. basis } \{\vec{e}_i\}}} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_d \end{bmatrix}$$

"Matrix element" $\boxed{A_{ij} = \vec{e}_i^{\,\dagger}\cdot(\hat{A}\vec{e}_j)}$