# Lecture 4: Foundations Continued

## Eigenvalues and diagonal representations

Given an operator $\hat{A}$, its "characteristic" equation

$$\hat{A}\,\underbrace{|a\rangle}_{\text{eigenvector}} = \underbrace{a}_{\text{eigenvalue}}\,|a\rangle$$

Here I am using a short hand of labeling the eigenvectors by their eigenvalues.

A Hermitian operator is self-adjoint $\hat{A} = \hat{A}^\dagger$

The operators have real eigenvalues

Proof: $\langle a|\hat{A}|a\rangle = a\,\langle a|a\rangle$

$$\Rightarrow \langle a|\hat{A}|a\rangle^* = \langle a|\hat{A}^\dagger|a\rangle = a^*\,\langle a|a\rangle^*$$

But $\hat{A} = \hat{A}^\dagger$ and $\langle a|a\rangle^* = \langle a|a\rangle$

$$\Rightarrow a = a^* \qquad \underline{a \text{ is real}}$$

Theorem: The eigenvectors associated with non-degenerate eigenvalues of a Hermitian operator are orthogonal

Proof: Consider $\langle a'|\hat{A}|a\rangle$ where $\hat{A}|a\rangle = a|a\rangle$, $\hat{A}|a'\rangle = a'|a\rangle$

$$\langle a'|\hat{A}|a\rangle = \langle a'|\left(\hat{A}|a\rangle\right) = a\,\langle a'|a\rangle$$

$$= \left(\langle a'|\hat{A}\right)|a\rangle = a'\,\langle a'|a\rangle$$

If $a \neq a' \Rightarrow \langle a'|a\rangle = 0$

The set of eigenvectors form a resolution of the identity (we won't prove this)

$$\hat{\mathbb{1}} = \sum_a |a\rangle\langle a|$$

- $\{|a\rangle\}$ = orthonormal basis

$$\hat{A} = \hat{\mathbb{1}}\hat{A}\hat{\mathbb{1}} = \sum_{a,a'} |a\rangle \underbrace{\langle a|\hat{A}|a'\rangle}_{a\,\delta_{aa'}} \langle a'| = \sum_a a\,|a\rangle\langle a|$$

Action of $\hat{A}$ = Project onto eigenvector $|a\rangle$ and multiply by eigenvalue $a$

Matrix representation of $\hat{A}$ in basis of eigenvectors

$$\langle a'|\hat{A}|a\rangle = a\,\delta_{a'a}$$

$$\hat{A} \doteq \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & \ddots \end{bmatrix}$$

Diagonal Matrix

Finding eigenvalues of $\hat{A}$ $\equiv$ "diagonalizing the matrix"

## Hilbert Space and Wave functions

A space of interest in quantum mechanics is that of "square integrable" complex function on space (1D, 2D, 3D)

In 1D we call this $\mathcal{L}_2(\mathbb{R})$

$$\psi(x) \in \mathcal{L}_2(\mathbb{R}^3) \Rightarrow \int dx\, |\psi(x)|^2 < \infty$$

This space is an <u>infinite</u> <u>dimensional</u> vector space with the inner product

$$\langle \phi | \psi \rangle = \int dx\, \phi^*(x)\, \psi(x)$$

$\exists$ bases of orthonormal functions (countably $\infty$)

$$\{ u_n(x) \} \qquad \int dx\, u_n^*(x)\, u_m(x) = \delta_{nm}$$

Any $\psi(x) \in \mathcal{L}_2(\mathbb{R})$, $\begin{cases} \psi(x) = \sum_{n=0}^{\infty} c_n u_n(x) \\ c_n = \int dx\, u_n^*(x)\, \psi(x) \end{cases}$

"Abstractify" $\psi(x) \doteq |\psi\rangle$ ← ket in Hilbert space

$$\Rightarrow |\psi\rangle = \sum_n |u_n\rangle \underbrace{\langle u_n | \psi \rangle}_{c_n}$$

$c_n$ ← infinite dimensional column vector

$$\hat{A}|\psi\rangle = |\phi\rangle \Rightarrow \sum_m \langle u_n|\hat{A}|u_m\rangle\langle u_m|\psi\rangle = \langle u_n|\phi\rangle$$

$$A_{nm} = \infty \times \infty \text{ matrix}$$

The matricies for $\infty$ dimensions are not particularly useful. However, quite often we are only interesded in a finite dimensional subspace, as we will see.

Continuous representions

Some operators have a continuous spectrum

E.g. position $\hat{x}$, eigenvalues $x$ (real line) $-\infty < x < \infty$

$\hat{p}$, eigenvalues $-\infty < p < \infty$

Technically their eigenvectors are not in the Hilbert space $\mathcal{L}_2(\mathbb{R})$. However they live in the "dual space", and can be used to find representations

$$\psi(x) = \langle x|\psi\rangle$$ wave function in "position space"

$$\tilde{\psi}(p) = \langle p|\psi\rangle$$ wave function in "momentum space"

Resolution of the identity:

$$\hat{\mathbb{1}} = \int_{-\infty}^{\infty} dx \; |x\rangle\langle x| = \int_{\infty}^{\infty} dp \; |p\rangle\langle p|$$

$$\Rightarrow \langle\phi|\psi\rangle = \int_{-\infty}^{\infty} dx \; \langle\phi|x\rangle\langle x|\psi\rangle = \int_{\infty} dx \; \phi^*(x)\,\psi(x)$$

insert complete set

$$= \int_{-\infty}^{\infty} dp \; \langle\phi|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp \; \tilde{\phi}^*(p)\,\tilde{\psi}(p)$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \; |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx \; \psi(x)\,|x\rangle$$

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \; \psi(x)\,\langle p|x\rangle$$

Aside: $\langle p|x\rangle = \langle x|p\rangle^* = u_p^*(x)$

$$= \text{planewave} = \left(\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\right)^* = \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}$$

$$\Longrightarrow \boxed{\tilde{\psi}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \, e^{-ipx/\hbar} \; \psi(x)}$$

Fourier transform.

This is an example of "change of basis

## Operators position and momentum space

In the continuous bases $\{|x\rangle\}$ and $\{|p\rangle\}$ operators are no longer represented as matricies but instead differential operators etc.

Consider: $\hat{x}|\psi\rangle$ and $\hat{p}|\psi\rangle$

position representation

$$\langle x|\hat{x}|\psi\rangle = x\langle x|\psi\rangle = x\,\psi(x)$$

eigenvector of $\hat{x}$

$$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle = \frac{\hbar}{i}\frac{\partial \psi}{\partial x}$$

we won't prove this

momentum representation

$$\langle p|\hat{x}|\psi\rangle = \frac{\hbar}{-i}\frac{\partial}{\partial p}\langle p|\psi\rangle = \frac{\hbar}{-i}\frac{\partial \tilde{\psi}}{\partial p}$$

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle = p\,\tilde{\psi}(p)$$

eigenvector

"Matrix elements" with wave functions

$$\langle \phi | \hat{A} | \psi \rangle = \int dx\, \phi^*(x) \left( \hat{A}_x \psi(x) \right)$$

$\uparrow$ position rep of $\hat{A}$

or since $\hat{A}$ acts on bra as $\hat{A}^\dagger$

$$\langle \phi | \hat{A} | \psi \rangle = \int dx \left( \hat{A}_x^\dagger \phi(x) \right)^* \psi(x)$$

$\hat{A}$ is Hermitian if

$$\int dx\, \phi^*(x) \left( \hat{A}_x \psi(x) \right) = \int dx \left( \hat{A}_x \phi(x) \right)^* \psi(x)$$

as we defined last semester.