

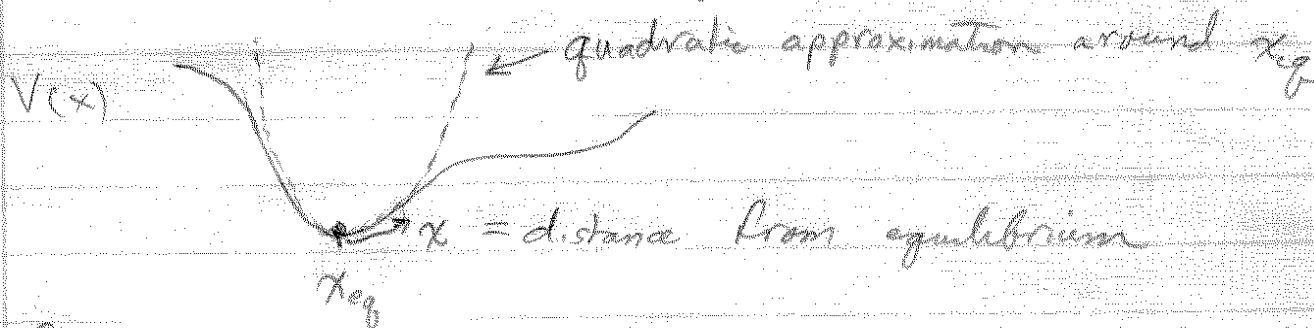
Physics 492 - Quantum II

Lecture 5: Simple Harmonic Oscillator (Introduction)

The simple harmonic oscillator (SHO) is one of (if not the) most important problems in physics. It describes the oscillation of a system slightly displaced from a point of stable equilibrium. The ubiquitous phenomenon of resonance is easily seen for a SHO. The SHO is central to understanding such systems as

- Molecular vibrations
- Crystal phonons
- Quantum fields (normal modes)

If there is a stable equilibrium point x_{eq}



Classical picture

Hook's law (restoring force)

$$F = -kx$$

(linear)

$$F = m\ddot{x} = -kx \Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

$$\text{Let } \omega = \sqrt{k/m}$$

Solution $\Rightarrow x(t) = A \cos(\omega t + \phi) = x(0) \cos \omega t + \frac{\dot{x}(0)}{\omega} \sin \omega t$

Hamiltonian Formulation:

$$H = \frac{p^2}{2m} + V(x)$$

$$V(x) = \frac{1}{2} K x^2 = \frac{1}{2} m \omega^2 x^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\Rightarrow \ddot{x} = \frac{\dot{p}}{m} = -\omega^2 x \quad \text{SHO} \quad \checkmark$$

Make dimensionless:

Let us suppose there is a characteristic energy scale E_c

$$\Rightarrow \text{Characteristic momentum: } \frac{p_c^2}{2m} = E_c \Rightarrow p_c = \sqrt{2mE_c}$$

$$\text{Characteristic position: } \frac{1}{2} m \omega^2 x_c^2 = E_c \Rightarrow x_c = \sqrt{\frac{2E_c}{m\omega^2}}$$

$$\text{Let } h = \frac{H}{E_c}, \quad P = \frac{p}{p_c}, \quad X = \frac{x}{x_c}$$

$$\Rightarrow \boxed{h = X^2 + P^2}$$

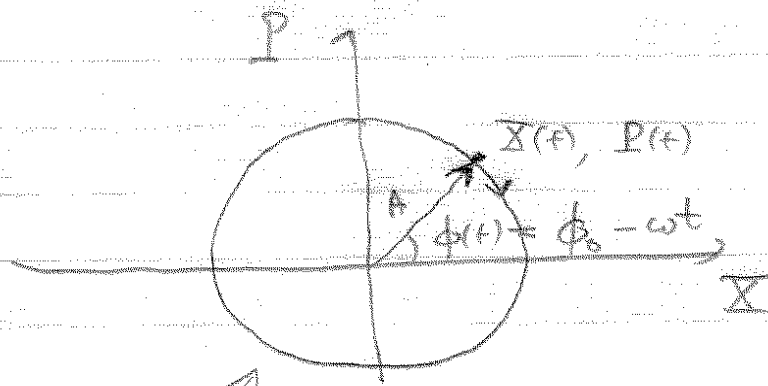
X and P look the same

$$\text{Hamilton's Equations: } \dot{X} = \omega P, \quad \dot{P} = -\omega X$$

$$\Rightarrow X(t) = X(0) \cos \omega t + P(0) \sin \omega t$$

$$P(t) = P(0) \sin \omega t - X(0) \cos \omega t$$

Phase Space



$h = \text{constant} = \text{circle}$

$$X(t) = A \cos(\phi_0 - \omega t) = \text{Re} \left(\underbrace{A e^{i\phi_0}}_{\alpha_0} e^{-i\omega t} \right)$$

$$\text{Re}(\alpha_0) = A \cos \phi_0 = X(0)$$

$$\text{Im}(\alpha_0) = A \sin \phi_0 = P(0)$$

$\alpha(t)$

$$\begin{aligned} \text{Let } \alpha(t) &= X(t) + iP(t) = A e^{i\phi(t)} \\ &= \alpha(0) e^{-i\omega t} = \underline{\text{phasor}} \end{aligned}$$

$$\text{Note } h = X^2 + P^2 = |\alpha(t)|^2$$

$$h = \alpha^* \alpha \quad (\text{amplitude squared} \\ = \text{energy})$$

The phasor picture is very useful for classical oscillators.

We will see its analogue now

in the quantum theory

Quantized SHO

To look at the Quantum theory we let x and p become operators

$$x \rightarrow \hat{x} \quad p \rightarrow \hat{p} \quad \text{with} \quad [\hat{x}, \hat{p}] = i\hbar$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

The new constant \hbar provides a scale for the action from we obtain a natural scale for energy

$$E_c = \hbar\omega$$

$$\Rightarrow p_c = \sqrt{2m\hbar\omega} \quad x_c = \sqrt{\frac{2\hbar}{m\omega}}$$

$$\text{Let} \quad \hat{X} = \frac{\hat{x}}{x_c} \quad \hat{P} = \frac{\hat{p}}{p_c}$$

$$[\hat{X}, \hat{P}] = \frac{1}{x_c p_c} [\hat{x}, \hat{p}] = \frac{i\hbar}{2\hbar} = \frac{i}{2}$$

$$\hat{H} = \hbar\omega (\hat{X}^2 + \hat{P}^2)$$

Now let $\hat{a} \equiv \hat{X} + i\hat{P}$ (non-Hermitian operator)

$$\hat{a}^\dagger = \hat{X} - i\hat{P}$$

(Quantum analog of α and α^*)

$$\begin{aligned}
 [\hat{a}, \hat{a}^\dagger] &= [\hat{X} + i\hat{P}, \hat{X} - i\hat{P}] \\
 &= -i \underbrace{[\hat{X}, \hat{P}]}_{i/2} + i \underbrace{[\hat{P}, \hat{X}]}_{-i/2}
 \end{aligned}$$

$$\Rightarrow \boxed{[\hat{a}, \hat{a}^\dagger] = 1} \quad \text{"canonical commutation relation"}$$

Let's express the Hamiltonian, \hat{H} , \hat{P} in terms of \hat{a} and \hat{a}^\dagger

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2} \quad \hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{2i}$$

$$\Rightarrow \hat{X}^2 = \frac{1}{4} (\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$$

$$\hat{P}^2 = \frac{1}{4} (-\hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$$

$$\Rightarrow \hat{H} = \frac{\hbar\omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$$

Aside: $\hat{a}^\dagger \hat{a} = \hat{a} \hat{a}^\dagger + [\hat{a}^\dagger, \hat{a}] = \hat{a} \hat{a}^\dagger + 1$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2})}$$

This is a form that is one of the most familiar in quantum mechanics

The "Number operator"

Let $\hat{N} \equiv \hat{a}^\dagger \hat{a}$ (Quantum analogue of $\alpha \alpha^*$)

$$\Rightarrow \hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$$

\Rightarrow Eigenvectors of \hat{N} are the energy eigenvectors (stationary state)

Commutation relations:

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger \underbrace{[\hat{a}, \hat{a}]}_{=0} + \underbrace{[\hat{a}^\dagger, \hat{a}]}_{=-1} \hat{a}$$

$$[\hat{N}, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger \underbrace{[\hat{a}, \hat{a}^\dagger]}_{=1}$$

$$\therefore \begin{cases} [\hat{N}, \hat{a}] = -\hat{a} \\ [\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger \end{cases}$$

(We will use these to find the energy eigenvalues algebraically)

Other properties of the number operator

\hat{N} is Hermitian

(Aside $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$)

$$\Rightarrow \hat{N}^\dagger = (\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger (\hat{a})^\dagger = \hat{a}^\dagger \hat{a} = \hat{N}$$

$\Rightarrow \hat{N}$ has real eigenvalues and orthogonal (if non-degenerate) eigenvectors

$$\hat{N} |\lambda\rangle = \lambda |\lambda\rangle$$

\uparrow real #

The eigenvalues are positive (non-negative)

Proof: Consider an arbitrary state

$$\begin{aligned} \langle \psi | \hat{N} | \psi \rangle &= \langle \psi | \hat{a}^\dagger \hat{a} | \psi \rangle = \langle \phi | \phi \rangle \\ &= \|\phi\|^2 \geq 0 \quad \text{where } |\phi\rangle = \hat{a} |\psi\rangle \end{aligned}$$

$\Rightarrow \hat{N}$ is a "positive operator"

\Rightarrow It's eigenvalues are non-negative

(see homework)

\Rightarrow Eigenvalues of $\hat{H} \geq \frac{\hbar \omega}{2}$ (zero point)