

Physics 492 - Quantum II

Lecture 6: Solutions to the T.I.S.E for SHO

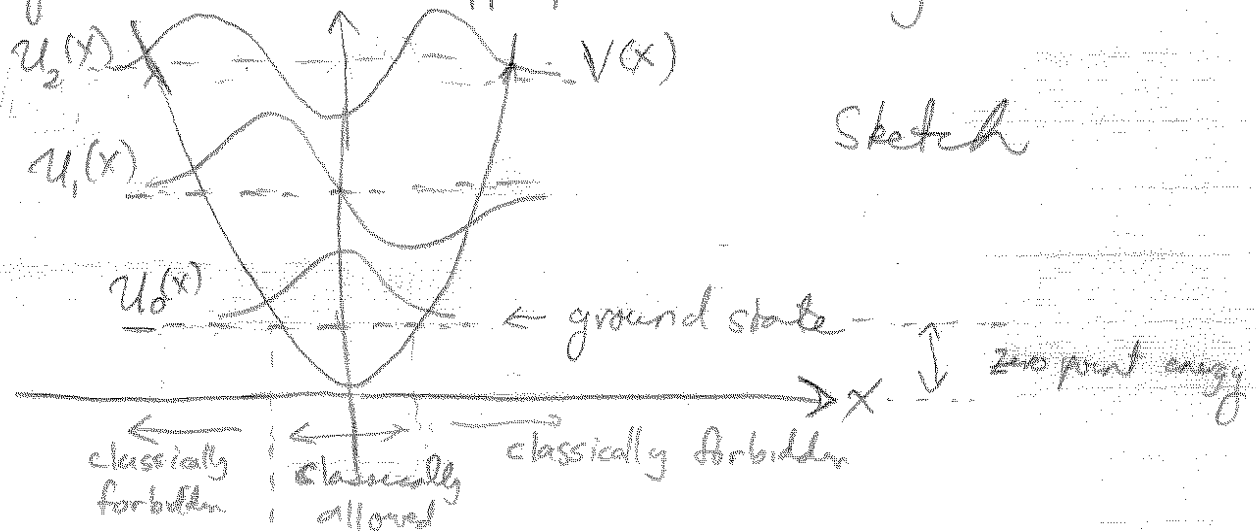
We seek solutions to the Time Independent Schrödinger Equation for the Simple Harmonic Oscillator.

$$\hat{H} |u_E\rangle = E |u_E\rangle$$

$$\text{where } \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

One approach is to solve the differential equation $\left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) u(x) = E u(x)$

subject to the appropriate boundary conditions



From our studies last semester we can sketch the energy eigenfunctions. Note $V(x) = V(-x)$
→ Energy eigenfunctions are parity eigenfunctions and alternate even/odd. Each excited state has one more node.

Like the infinite square well, the harmonic well is infinitely high

⇒ Only bound state

⇒ Spectrum of \hat{H} is discrete (non-degenerate)

⇒ Hilbert space $L_2(\mathbb{R})$ spanned by eigenfunctions of \hat{H}

Instead of using differential equation we can use the algebraic methods.

Let us write $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$
where $\hat{N} = \hat{a}^\dagger \hat{a}$

Recall $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$

\hat{N} is a "positive operator" ⇒ eigenvalues are ≥ 0

Let $\hat{N} |u_\lambda\rangle = \lambda |u_\lambda\rangle \quad \lambda \geq 0$

⇒ $\hat{H} |u_\lambda\rangle = \underbrace{\hbar\omega(\lambda + \frac{1}{2})}_{E_\lambda} |u_\lambda\rangle$

We thus seek λ

Lemma: Let $|v_\lambda\rangle = \hat{a}|u_\lambda\rangle$

$$\Rightarrow \hat{N}|v_\lambda\rangle = (\lambda-1)|u_\lambda\rangle$$

i.e. $\hat{a}|u_\lambda\rangle$ eigenvector of \hat{N} , eigenvalue $\lambda-1$

Proof: $\hat{N}|v_\lambda\rangle = \hat{N}\hat{a}|u_\lambda\rangle$

$$= (\hat{a}\hat{N} + [\hat{N}, \hat{a}])|u_\lambda\rangle$$

$$= \hat{a}\hat{N}|u_\lambda\rangle - \hat{a}|u_\lambda\rangle =$$

$$= \lambda|u_\lambda\rangle - \hat{a}|u_\lambda\rangle$$

$$\Rightarrow \hat{N}|v_\lambda\rangle = (\lambda-1)\hat{a}|u_\lambda\rangle = (\lambda-1)|v_\lambda\rangle$$

q.e.d.

Lemma: Let $|w_\lambda\rangle = \hat{a}^\dagger|u_\lambda\rangle$

$$\Rightarrow \hat{N}|w_\lambda\rangle = (\lambda+1)|u_\lambda\rangle$$

Proof: $\hat{N}|w_\lambda\rangle = \hat{N}\hat{a}^\dagger|u_\lambda\rangle = (\hat{a}^\dagger\hat{N} + [\hat{N}, \hat{a}^\dagger])|u_\lambda\rangle$

$$= (\hat{a}^\dagger\hat{N} + \hat{a}^\dagger)|u_\lambda\rangle$$

$$= (\lambda+1)\hat{a}^\dagger|u_\lambda\rangle$$

$$= (\lambda+1)|w_\lambda\rangle \quad \text{q.e.d.}$$

Final lemma: The eigenvalues of \hat{N} are
non-negative integers

$$\lambda = 0, 1, 2, \dots, \infty$$

Proof: Suppose λ were not an integer

$$\text{Consider } |\Phi\rangle = \hat{a}^m |u_\lambda\rangle$$

$$\Rightarrow \hat{N}|\Phi\rangle = (\lambda - m)|u_\lambda\rangle$$

$\Rightarrow \exists m$ such that $\lambda - m < 0$
contradiction since \hat{N} is positive

If $\lambda = \text{integer} = n$

$$\Rightarrow \text{When } m = n \quad \hat{N} \hat{a}^n |u_n\rangle = 0$$

$$\text{Let } |u_0\rangle = \hat{a}^{n-1} |u_n\rangle$$

$$\Rightarrow \hat{N}(\hat{a}|u_0\rangle) = 0$$

However since \hat{N} is positive

$$\Rightarrow \hat{a}|u_0\rangle = 0$$

$\Rightarrow \exists$ a ground state defined by

$\Rightarrow \lambda$ must be an integer

Thus we now know why \hat{N} is referred to as the "number operator"

$$\hat{N} |u_n\rangle = n |u_n\rangle$$

$$n = 0, 1, 2, 3, \dots$$

where $\hat{a} |u_0\rangle = 0$

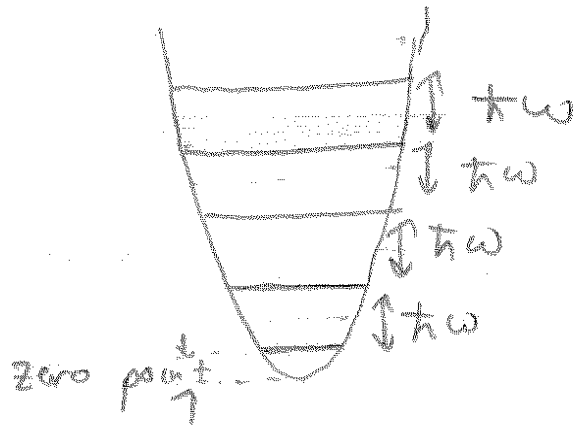
$$\Rightarrow \hat{a} = \text{"annihilation operator"}$$

Energy eigenvalues

Let $|n\rangle \equiv |u_n\rangle$

$$\hat{H} |n\rangle = E_n |n\rangle$$

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right) = \left(\frac{\hbar\omega}{2}, \frac{3\hbar\omega}{2}, \frac{5\hbar\omega}{2}, \dots\right)$$



Unique feature of SHO:

Equally spaced energy levels

Eigenvectors: $\left. \begin{array}{l} \text{Raising (creation)} \\ \text{Lowering (annihilation)} \end{array} \right\} \text{ operators}$

Consider: $\hat{a}|n\rangle = c_n|n-1\rangle$

$$\begin{aligned} \langle n|\hat{a}^\dagger(\hat{a}|n\rangle) &= |c_n|^2 \underbrace{\langle n-1|n-1\rangle}_{=1} \\ &= \langle n|\hat{a}^\dagger\hat{a}|n\rangle \\ &= \langle n|\hat{N}|n\rangle = n \end{aligned}$$

$\therefore |c_n| = \sqrt{n}$ (take real $\Rightarrow c_n = \sqrt{n}$)

$\Rightarrow \boxed{\hat{a}|n\rangle = \sqrt{n}|n-1\rangle}$

\hat{a} = lowering operator = annihilation

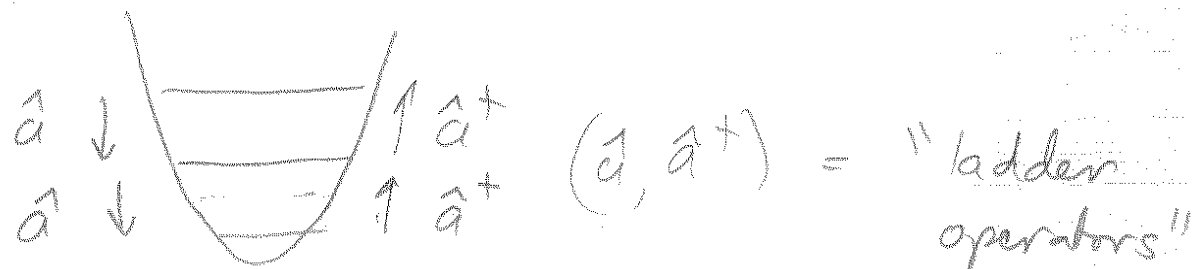
Similarly $\hat{a}^\dagger|n\rangle = k_n|n+1\rangle$

$$\begin{aligned} \langle n|\hat{a}\hat{a}^\dagger|n\rangle &= \langle n|(\hat{a}^\dagger\hat{a} + 1)|n\rangle \\ &= |k_n|^2 \langle n+1|n+1\rangle = (n+1) \langle n|n\rangle \end{aligned}$$

$\Rightarrow |k_n| = \sqrt{n+1}$, choose $k_n = \sqrt{n+1}$

$\Rightarrow \boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle}$

\hat{a}^\dagger = raising operator = creation



Finally, given $\hat{a}^+ |n\rangle = \sqrt{n+1} |n+1\rangle$

$$\Rightarrow \hat{a}^+ |0\rangle = |1\rangle$$

$$(\hat{a}^+)^2 |0\rangle = \hat{a}^+ |1\rangle = \sqrt{2} |2\rangle$$

$$(\hat{a}^+)^3 |0\rangle = \hat{a}^+ \sqrt{2} |2\rangle = \sqrt{3 \cdot 2} |3\rangle$$

$$\Rightarrow (\hat{a}^+)^n |0\rangle = \sqrt{n!} |n\rangle$$

\Rightarrow We can express the n^{th} level in terms of the ground state

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle$$