Physics 492 - Quantum II

Lecture 8: Angular Momentum, Algebraic Approach

Orbital Angular Momentum (Review)

The angular momentum associated with the motion of a ~~particular~~ particle is described by the operator

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

whose components satisfy the commutation relations

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

Because the vector components don't commute they don't share a common complete set of eigenstates. Thus there is no ~~state~~ set of states with a definite angular momentum vector.

On the other hand, $\hat{L}^2 \equiv \hat{\vec{L}} \cdot \hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

(magnitude squared)

$$[\hat{L}^2, \hat{L}_i] = 0 \quad \text{(any component } \hat{L}_i\text{)}$$

$\Rightarrow \exists$ eigenstates which are common eigenvectors of $\hat{L}^2$ and any <u>one</u> component $\hat{L}_i$.

Typically, we choose this component to be $\hat{L}_z$.

Eigenstates, wave mechanics approach

$$\hat{\vec{p}} \doteq \frac{\hbar}{i}\vec{\nabla} \Rightarrow \hat{\vec{L}} \doteq \frac{\hbar}{i}\vec{r}\times\vec{\nabla}$$

$$\Rightarrow \hat{p}^2 = -\hbar^2\nabla^2 = \hat{p}_r^2 + \frac{1}{r^2}\hat{L}^2$$

$$\hat{p}_r^2 \doteq -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ )$$

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}\ ) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

Eigenfunctions: $\underline{Y_\ell^m(\theta,\phi)}$ (spherical harmonics)

Seperation of variables $Y_\ell^m(\theta,\phi) = \Theta_\ell^m(\theta)\,\Phi_m(\phi)$

$$\underbrace{\hat{L}_z\Phi_m(\phi) = \hbar m\,\Phi_m(\phi)}\ \Rightarrow\ \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

Single valued wave function $\Rightarrow \Phi_m(\phi+2\pi) = \Phi_m(\phi)$

$\Longrightarrow$ <u>m is integer</u>

$\Rightarrow \Theta_\ell^m(\theta) = C_\ell\,P_\ell^m(\cos\theta)$ ← Legendre Polynomial

$$\boxed{\hat{L}^2\,Y_\ell^m(\theta,\phi) = \hbar^2\,\ell(\ell+1)\,Y_\ell^m(\theta,\phi)}$$

Degeneracy
- $2\ell+1$ different m values
- $-\ell \le m \le \ell$
- ℓ, integer

Algebraic Solution.

As we accomplished for the SHO, we can solve the eigenvalue problem for angular momentum in quantum mechanics using solely operator algebra.

In the abstract we define $\hat{\vec{J}}$ as "angular momentum" satisfying the commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \quad \text{and cyclic permutations}$$

$$[\hat{J}^2, \hat{J}_i] = 0$$

Define dimensionless variable $\hat{\vec{j}} \equiv \frac{\hat{\vec{J}}}{\hbar}$

$$[\hat{j}_x, \hat{j}_y] = i\hat{j}_z \quad \text{etc.}$$

Look for common eigenvectors of

$$\hat{j}^2 = \hat{j}_x^2 + \hat{j}_y^2 + \hat{j}_z^2 \quad \text{and} \quad \hat{j}_z$$

$$\hat{j}^2 |\lambda, m\rangle = \lambda |\lambda, m\rangle \quad , \quad \lambda > 0$$

$$\hat{j}_z |\lambda, m\rangle = m |\lambda, m\rangle$$

Note: $\hat{j}_x^2 + \hat{j}_y^2 = \hat{j}^2 - \hat{j}_z^2$ is a <u>positive operator</u>

$$\Rightarrow \langle\lambda, m| \hat{j}_x^2 + \hat{j}_y^2 |\lambda, m\rangle = \lambda^2 - m^2 \geq 0$$

$$\Rightarrow \quad -\lambda \leq m \leq \lambda$$

Recall, for the SHO $H = X^2 + P^2$ (dimensionless)

we "factorized" $X^2 + P^2 = (X + iP)(X - iP)$

$$\hat{a} \equiv \hat{X} + i\hat{P}, \quad \hat{a}^\dagger \equiv \hat{X} - i\hat{P} \quad \text{"ladder operators"}$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

For angular momentum, we define

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y, \quad \hat{J}_- = \hat{J}_x - i\hat{J}_y = (\hat{J}_+)^\dagger$$

$$[\hat{J}_+, \hat{J}_-] = 2i\underbrace{[\hat{J}_x, \hat{J}_y]}_{i\hat{J}_z} = 2\hat{J}_z$$

$$[\hat{J}_z, \hat{J}_\pm] = \underbrace{[\hat{J}_z, \hat{J}_x]}_{i\hat{J}_y} \pm i\underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hat{J}_x} = \pm\hat{J}_x + i\hat{J}_y$$

$$= \pm(\hat{J}_x \pm i\hat{J}_y) = \pm\hat{J}_\pm$$

$$\Rightarrow \hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \underbrace{\hat{J}_x^2 + \hat{J}_y^2}_{\hat{J}^2 - \hat{J}_z^2} - i\underbrace{[\hat{J}_x, \hat{J}_y]}_{i\hat{J}_z}$$

$$\Rightarrow \hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z(\hat{J}_z - 1)$$

Similarly $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z(\hat{J}_z + 1)$

<u>Note</u>: $\lambda^2 - m^2 > m \Rightarrow -\lambda < m < \lambda$

Lemma: $\left\{\begin{array}{l} \hat{J}_\pm \text{ are "ladder operators"} \\ \hat{J}_z \left(\hat{J}_\pm |\lambda, m\rangle\right) = (m \pm 1) \left(\hat{J}_\pm |\lambda, m\rangle\right) \\ \text{i.e. } \hat{J}_\pm |\lambda, m\rangle = c_\pm |\lambda, m \pm 1\rangle \end{array}\right.$

Proof: $\hat{J}_z \hat{J}_\pm |\lambda, m\rangle = \left(\hat{J}_\pm \hat{J}_z + \underbrace{[\hat{J}_z, \hat{J}_\pm]}_{\pm \hat{J}_\pm}\right) |\lambda, m\rangle$

$= (m \pm 1) \hat{J}_\pm |\lambda, m\rangle$

q.e.d.

Lemma: There exists a maximum and minimum value of m for a given λ, where $m_{min} = -m_{max}$.

We saw $-\lambda < m < \lambda \Rightarrow \exists \; m_{max}, m_{min}$

$\Rightarrow \hat{J}_+ |\lambda, m_{max}\rangle = 0 \qquad \hat{J}_- |\lambda, m_{min}\rangle = 0$

Thus $\langle \lambda, m_{max} | \hat{J}_+^\dagger \hat{J}_+ | \lambda, m_{max}\rangle = \langle \lambda, m_{max} | \hat{J}_- \hat{J}_+ | \lambda, m_{max}\rangle = 0$

$\Rightarrow \langle \lambda, m_{max} | \hat{J}^2 - \hat{J}_z(\hat{J}_z + 1) | \lambda, m_{max}\rangle = 0$

$\therefore \lambda - m_{max}(m_{max} + 1) = 0$

$\Rightarrow \lambda = m_{max}(m_{max} + 1)$

Similarly $\langle \lambda, m_{min} | \hat{J}_-^\dagger \hat{J}_- | \lambda, m_{min} \rangle = \langle \lambda, m_{min} | \hat{J}_+ \hat{J}_- | \lambda\, m_{min}$

$$\Rightarrow \langle \lambda, m_{min} | \hat{J}^2 - \hat{J}_z (\hat{J}_z - 1) | \lambda, m_{min} \rangle = 0$$

$$\therefore \quad \lambda - m_{min}(m_{min} - 1) = 0$$

$$\Rightarrow \lambda = m_{min}(m_{min} - 1)$$

$$\therefore \quad m_{max}(m_{max} + 1) = m_{min}(m_{min} - 1)$$

$$\Rightarrow m_{max} = -m_{min} \qquad \text{q.e.d.}$$

Thus we <u>define</u> $j = m_{max} \Rightarrow \lambda = j(j+1)$

$$\Rightarrow \boxed{\begin{array}{l} \hat{J}^2 |j, m\rangle = j(j+1) |j, m\rangle \\ \hat{J}_z |j, m\rangle = m |j, m\rangle \\ \quad -j \le m \le j \quad \text{in integer steps} \\ \qquad 2j+1 \quad \text{different } m \text{ values for given } j \end{array}}$$

We have thus solved for the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ with purely operator algebra.

Raising and lowering operators:

$$\hat{J}_+ |j, m\rangle \stackrel{?}{=} c\, |j, m+1\rangle$$

$$\Rightarrow \langle j,m|\hat{J}_- \hat{J}_+|j,m\rangle = c^* c \underbrace{\langle j, m+1 | j, m+1\rangle}_{\equiv 1}$$

$$\hat{J}^2 - \hat{J}_z(\hat{J}_z + 1)$$

$$\Rightarrow |c|^2 = j(j+1) - m(m+1)$$

Phase convention, choose real $\Rightarrow$ $c = \sqrt{j(j+1) - m(m+1)}$

$$\boxed{\begin{array}{l} \Rightarrow \hat{J}_+ |j,m\rangle = \sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle \\ \text{Similarly } \hat{J}_- |j,m\rangle = \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle \end{array}}$$

<u>Orbital vs. Spin Angular Momentum</u>

When we found the spherical harmonic solutions, continuity of the wave function required <u>l = integer</u>.

However, the algebraic solution requires
$2j + 1$ = integer

$\Rightarrow$ Two possibilities $\begin{cases} j = \text{integer} \\ j = \text{half-integer} \end{cases}$

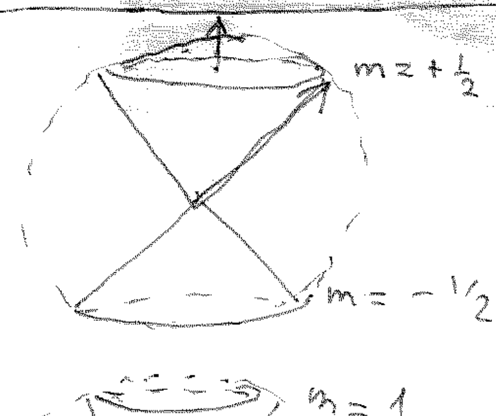
The half-integer solutions do not correspond to orbital angular momentum. They are possible solutions corresponding to an intrinsic angular momentum associated with elementary particles. We call this type of angular momentum spin. Note: Spin angular momentum can be either ½-integer or whole-integer. This divides the particles into two classes

half-integer:	fermions
whole-integer:	bosons

⇐ categories of spin angular momentum

"Space quantization" (Vector diagram)

$j = 1/2$



$j = 1$