Physics 412 - Quantum II

Lecture 9: Matrix Representations for Angular Mom.

Last time we found for angular momentum operators satisfying $[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$ (and cyclic permutations)

$[\hat{J}^2, \hat{J}_i] = 0$

$\exists$ common eigenstates of $\hat{J}^2$ and $\hat{J}_z$

$$\hat{J}^2 |j, m\rangle = j(j+1)\, |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = m|j, m\rangle$$

where $-j \leq m \leq j$ in integer step
($2j+1$ different m's for given j)

Raising and lowering operators: $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z \qquad [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm$$

$$\hat{J}_+|j, m\rangle = \sqrt{j(j+1) - m(m+1)}\; |j, m+1\rangle$$

$$\hat{J}_-|j, m\rangle = \sqrt{j(j+1) - m(m-1)}\; |j, m-1\rangle$$

Note: $\hat{J}_+ |j, m=j\rangle = 0$, $\hat{J}_-|j, m=-j\rangle = 0$

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \qquad \hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

We take the set $\{ |j, m\rangle \mid j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$
$-j \leq m \leq j \}$
to be the "standard basis"

<u>Note:</u> The sub-set of $2j+1$ vectors for a given j span a <u>subspace</u> of the total Hilbert space

Matrix elements:

$$\langle j' m' | \hat{J}^2 | j\, m\rangle = j(j+1)\, \delta_{jj'}\, \delta_{mm'}$$

$$\langle j' m' | \hat{J}_z | j, m\rangle = m\, \delta_{jj'}\, \delta_{mm'}$$

$$\langle j' m' | \hat{J}_+ | j, m\rangle = \sqrt{j(j+1) - m(m+1)}\, \delta_{jj'}\, \delta_{m', m+1}$$

$$\langle j' m' | \hat{J}_- | j, m\rangle = \sqrt{j(j+1) - m(m-1)}\, \delta_{jj'}\, \delta_{m', m-1}$$

$$\langle j' m' | \hat{J}_x | j, m\rangle = \frac{\sqrt{j(j+1) - m(m+1)}}{2}\, \delta_{jj'} \left(\delta_{m', m+1} + \delta_{m'+1, m}\right)$$

$$\langle j', m' | \hat{J}_y | j, m\rangle = -i\frac{\sqrt{j(j+1) - m(m+1)}}{2}\, \delta_{jj'} \left(\delta_{m', m+1} - \delta_{m'+1, m}\right)$$

<u>Note:</u> No off-diagonal elements between different j values $\Longrightarrow$ If we order the basis so all m's for given j are together, ~~all~~ matrices are <u>block-diagonal</u>

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{1/2} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{3/2} \oplus \mathcal{H}_2 \oplus \cdots$$

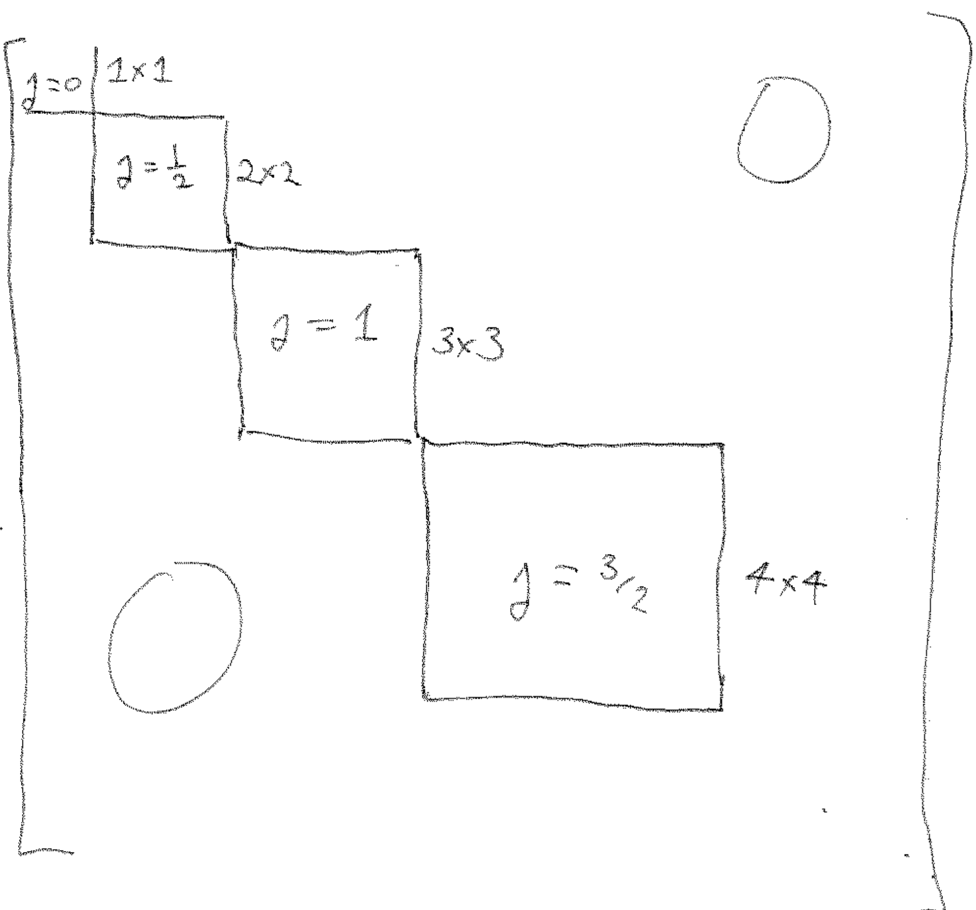
"Direct sum" of orthogonal subspaces

Basis $\Big\{ \{|0,0\rangle\}, \{|\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle\}, \{|1,1\rangle, |1,0\rangle, |1,-1\rangle\},$

$\{|\tfrac{3}{2},\tfrac{3}{2}\rangle, |\tfrac{3}{2},\tfrac{1}{2}\rangle, |\tfrac{3}{2},-\tfrac{1}{2}\rangle, |\tfrac{3}{2},-\tfrac{3}{2}\rangle\},$

$\{|2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle, |2,-2\rangle\}, \quad \cdots \Big\}$

$\hat{J}_i \doteq \hbar$



Decomposition of any $\hat{J}$ operator into orthogonal blocks

<u>Example:</u> $j = 1/2$ Basis $\{ |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \}$

Short hand $|\uparrow\rangle \equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle$ "spin-up"

$|\downarrow\rangle \equiv |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ "spin-down"

$$\hat{J}^2 |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1) \ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \tfrac{3}{4} \ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

$$\hat{J}_z |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm\tfrac{1}{2} \ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

$$\hat{J}_+ |\tfrac{1}{2}, +\tfrac{1}{2}\rangle = 0 \qquad \hat{J}_- |\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)} \ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\Rightarrow \hat{J}_- |\tfrac{1}{2}, +\tfrac{1}{2}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\hat{J}_+ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(-\tfrac{1}{2}+1)} \ |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$\Rightarrow \hat{J}_+ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad \hat{J}_- |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0$$

$$\Rightarrow \quad \hat{J}^2 \doteq \tfrac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \langle\uparrow| \\ \langle\downarrow| \end{matrix}$$

$$\hat{J}_z \doteq \hbar \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{matrix} \langle\uparrow| \\ \langle\downarrow| \end{matrix}$$

$$\hat{J}_+ \doteq \begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} \langle\uparrow| \\ \langle\downarrow| \end{matrix} \\ \begin{matrix} |\uparrow\rangle & |\downarrow\rangle \end{matrix} & \end{array} \qquad \hat{J}_- \doteq \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{matrix} \langle\uparrow| \\ \langle\downarrow| \end{matrix} \\ \begin{matrix} |\uparrow\rangle & |\downarrow\rangle \end{matrix} & \end{array}$$

$$\hat{J}_x = \hbar \left(\frac{\hat{J}_+ + \hat{J}_-}{2} \right) \doteq \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{J}_y = \hbar \left(\frac{\hat{J}_+ - \hat{J}_-}{2i} \right) \doteq \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Example $j=1$ Basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$

$$\hat{J}^2 |1,m\rangle = \sqrt{1(1+1)}\, |1,m\rangle = \sqrt{2}\, |1,m\rangle$$

$$\hat{J}_z |1,m\rangle = m\, |1,m\rangle$$

$$\hat{J}_+ |1,m\rangle = \sqrt{2-m(m+1)}\, |1,m+1\rangle$$

$$\Rightarrow \quad \hat{J}^2 \doteq \hbar^2\sqrt{2}\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \qquad \hat{J}_z \doteq \hbar\begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{bmatrix}$$

$$\hat{J}_+ \doteq \begin{array}{c} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \\ |1\rangle \quad |0\rangle \quad |-1\rangle \end{array} \begin{array}{l} \langle 1,1| \\ \langle 1,0| \\ \langle 1,-1| \\ \end{array}$$

$$\hat{J}_- = \hat{J}_+^\dagger \doteq \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$\hat{J}_x \doteq \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{J}_y \doteq \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Finding eigenvalues/vectors using matricies

Suppose we seek the eigenvectors of $\hat{J}_x$. Because we know this operator does not "connect" states with different magnitude j, there exist eigenvectors within a given subspace $\mathcal{H}_j$, spanned by the basis of dimension $2j+1$ $\{|j, m\rangle \mid -j \leq m \leq j\}$.

In that basis $\hat{J}_x$ is a $(2j+1) \times (2j+1)$ matrix.

e.g. $j = \frac{1}{2}$ $\qquad \hat{J}_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We thus seek the common eigenvectors of $\hat{J}^2$ and $\hat{J}_x$ with j-eigenvalue $j = \frac{1}{2}$

Denote $|j, m_x\rangle$ where $\hat{J}_x |j, m_x\rangle = m_x |j, m_x\rangle$

$$\Rightarrow \left(\hat{J}_x - m_x \hat{\mathbb{1}}\right)|j, m_x\rangle = 0$$

Expressed in the "standard basis" $\{|\uparrow\rangle = |\tfrac{1}{2}, m_z = \tfrac{1}{2}\rangle$
$|\downarrow\rangle = |\tfrac{1}{2}, m_z = -\tfrac{1}{2}\rangle$

$$\begin{bmatrix} -m_x & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -m_x \end{bmatrix}\begin{bmatrix} C_\uparrow \\ C_\downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Aside:</u> Theorem of linear alegbra.

Given matrix $\hat{M}$, vector $|v\rangle$

if $\hat{M}|v\rangle = 0$ and $|v\rangle \neq$ null vector

$\det[\hat{M}] = 0$ (rows are linearly dependent)

$\nwarrow$ <u>determinant</u>

Thus, the eigenvalues are the solutions to

$$\det(\hat{J}_x - m_x \hat{\mathbb{1}}) = 0$$

$\equiv$ <u>Characteristic equation</u>

= Polynomial of order d = dimension of matrix

For the example at hand

$$\det \begin{bmatrix} -m_x & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -m_x \end{bmatrix} = m_x^2 - \frac{\hbar^2}{4} = 0$$

(second order for $j = \frac{1}{2}$)

$$\Rightarrow \boxed{m_x = \pm\frac{\hbar}{2}}$$

These are the same <u>eigenvalues</u> as for $\hat{J}_z$. This makes physical sense since x, y, z arbitrary labels.

However, the <u>eigenvectors</u> of $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ are different. These operators don't commute.

Finding eigenvector $|j, m_x\rangle = \sum_{m_z=-j}^{j} c_{m_z} |j, m_z\rangle$

$$\begin{bmatrix} -m_x & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -m_x \end{bmatrix} \begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Two cases:

$$m_x = +\frac{\hbar}{2} \Rightarrow \frac{\hbar}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -c_\uparrow + c_\downarrow &= 0 \\ c_\uparrow - c_\downarrow &= 0 \end{aligned}\right\} \text{ same solution since linearly dependent}$$

$$\Rightarrow c_\uparrow = c_\downarrow$$

$$\Rightarrow |j=\tfrac{1}{2}, m_x = \tfrac{1}{2}\rangle = c_\uparrow\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

Not normalized

$$\langle j=\tfrac{1}{2}, m_x = \tfrac{1}{2} | j=\tfrac{1}{2}, m_x = \tfrac{1}{2}\rangle = 2|c_\uparrow|^2 = 1$$

$$\Rightarrow |c_\uparrow| = \frac{1}{\sqrt{2}} \quad \text{(choose real)}$$

$$\Rightarrow \boxed{|j=\tfrac{1}{2}, m_x = \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)}$$

Similarly $\boxed{|j=\tfrac{1}{2}, m_x = -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)}$